

Fermions are not Qubits

musings about fermionic mode entanglement

Seminar Talk, IQOQI Vienna

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Introduction

Motivation

- Fermions fundamental part of description of nature
 - essentially all (standard) matter consists of fermions
- Effects of QFT on curved spacetimes, e.g., Unruh/Hawking create mode entanglement
- Progress in simulations and electron quantum optics

Want to use fermionic modes in quantum information, but how?

"Fermions are like qubits . . . aren't they?"

- Pauli principle \Rightarrow two-level systems
 - fermionic mode k either occupied, $|1_k\rangle$, or not, $|0_k\rangle$
- For fixed number of modes \exists isomorphism:
 - Fock space \Rightarrow tensor product space
- QI: also need consistent notion of subsystems \Rightarrow Trouble¹

¹ Friis, Lee, Bruschi, *Fermionic mode entanglement in quantum information*, Phys. Rev. A 87, 022338 (2013).

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- Teleportation, Computation, etc.

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- Description of fermions: states on Fock spaces
 - Fermionic ambiguity—issues with partial trace
 - Consistency conditions for subsystems

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Fermions in a nutshell

Fermionic Fock space 101

Single-fermion wave functions ψ_m - modes - associate a_m, a_m^\dagger

anticommutation relations $\{a_m, a_n^\dagger\} = \delta_{mn}, \quad \{a_m, a_n\} = 0$

vacuum state $|0\rangle \in \mathcal{H}_{0-p}$, annihilated by all $a_m, \quad a_m|0\rangle = 0 \quad \forall m$

single-particle states: $|1_m\rangle = |\psi_m\rangle = a_m^\dagger |0\rangle \in \mathcal{H}_{1-p}$

two-particle states $\in S(\mathcal{H}_{1-p} \otimes \mathcal{H}_{1-p})$

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N -mode Fock space $\mathcal{F}_N(\mathcal{H}_{1-p})$

$$\mathcal{F}_N(\mathcal{H}_{1-p}) = \bigoplus_{m=0}^N \bar{S}\left(\mathcal{H}_{1-p}^{\otimes m}\right) = \mathcal{H}_{0-p} \oplus \mathcal{H}_{1-p} \oplus \bar{S}\left(\mathcal{H}_{1-p} \otimes \mathcal{H}_{1-p}\right) \oplus \dots$$

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Density operators on the Fock space

General pure state on the Fock space

$$|\psi\rangle = \mu_0 |0\rangle + \sum_{i \neq 0} \mu_i |1_i\rangle + \sum_{i,j} \mu_{ij} |1_i, 1_j\rangle + \dots$$

Mixed states \Rightarrow convex sums of projectors

Example: two-mode mixed state

$$\begin{aligned}\rho_{kk'} = & \alpha_1 |0\rangle\langle 0| + \alpha_2 a_{k'}^\dagger |0\rangle\langle 0| a_{k'} + \alpha_3 a_k^\dagger |0\rangle\langle 0| a_k \\ & + \alpha_4 a_k^\dagger a_{k'}^\dagger |0\rangle\langle 0| a_{k'} a_k + (\beta_1 |0\rangle\langle 0| a_{k'} + \beta_2 |0\rangle\langle 0| a_k \\ & + \beta_3 |0\rangle\langle 0| a_{k'} a_k + \beta_4 a_{k'}^\dagger |0\rangle\langle 0| a_k + \beta_5 a_{k'}^\dagger |0\rangle\langle 0| a_{k'} a_k \\ & + \beta_6 a_k^\dagger |0\rangle\langle 0| a_{k'} a_k + \text{H.c.})\end{aligned}$$

Partial Trace: $\text{Tr}_{k'}(\rho_{kk'})$

Density operators on the Fock space

General pure state on the Fock space

$$|\psi\rangle = \mu_0 |0\rangle + \sum_{i \neq 0} \mu_i |1_i\rangle + \sum_{i,j} \mu_{ij} |1_i, 1_j\rangle + \dots$$

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$$\begin{aligned}\rho_{kk'} &= \alpha_1 |0\rangle\langle 0| + \alpha_2 a_{k'}^\dagger |0\rangle\langle 0| a_{k'} + \color{blue}{\alpha_3 a_k^\dagger |0\rangle\langle 0| a_k} \\ &\quad + \color{blue}{\alpha_4 a_k^\dagger a_{k'}^\dagger |0\rangle\langle 0| a_{k'} a_k} + (\beta_1 |0\rangle\langle 0| a_{k'} + \beta_2 |0\rangle\langle 0| a_k \\ &\quad + \beta_3 |0\rangle\langle 0| a_{k'} a_k + \beta_4 a_{k'}^\dagger |0\rangle\langle 0| a_k + \beta_5 a_{k'}^\dagger |0\rangle\langle 0| a_{k'} a_k \\ &\quad + \beta_6 a_k^\dagger |0\rangle\langle 0| a_{k'} a_k + \text{H.c.})\end{aligned}$$

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The fermionic ambiguity

Partial traces appear ambiguous

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Signs are fixed if a particular mapping to tensor product is chosen

But not in the right way!

Consistency conditions¹

Expectation values of local observables invariant

$$\langle \mathcal{O}(k) \rangle_{\rho_{kk'}} = \langle \mathcal{O}(k) \rangle_{\text{Tr}_{k'}(\rho_{kk'})}$$

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Consistent mappings to qubits (?)

Practical consequence of consistency conditions

Partial tracing “inside-out”, e.g., $\text{Tr}_k: a_k^\dagger |0\rangle\langle 0| a_{k'} a_k$

anticommute towards vacuum projector

remove operators

Isomorphisms between n fermionic modes and n qubits

mappings from \mathcal{F}_N to $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$

e.g., $a_k^\dagger |0\rangle\langle 0| a_{k'} a_k \mapsto |1\rangle\langle 1|_k \otimes |0\rangle\langle 1|_{k'}$

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Role of superselection rules

In general consistent mapping should satisfy

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Superselection rules restrict allowed density operator elements

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2 modes: Yes, this is enough, problematic elements disappear

3 modes: No, some problematic elements remain

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So what are the superselection rules good for?

Suppose there were no superselection rules...

pure state of two modes k and k' prime:

$$|\psi\rangle = \mu_0 |0\rangle + \mu_k |1_k\rangle + \mu_{k'} |1_{k'}\rangle + \mu_{kk'} |1_k, 1_{k'}\rangle$$

reduced states:

$$\begin{aligned}\rho_k = & (|\mu_0|^2 + |\mu_k|^2) |0\rangle\langle 0| + (|\mu_k|^2 + |\mu_{kk'}|^2) |1_k\rangle\langle 1_k| \\ & + [(\mu_0\mu_k^* + \mu_k\mu_{kk'}^*) |0\rangle\langle 1_k| + H.c.] \end{aligned}$$

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e.g., for $\mu_0 = \mu_k = \mu_{k'} = \mu_{kk'} = \frac{1}{2}$: ρ_k pure, $\rho_{k'}$ maximally mixed

⇒ asymmetric pure state marginals^{2,3}

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So what are the superselection rules good for?

Suppose there were no superselection rules...

pure state of two modes k and k' prime:

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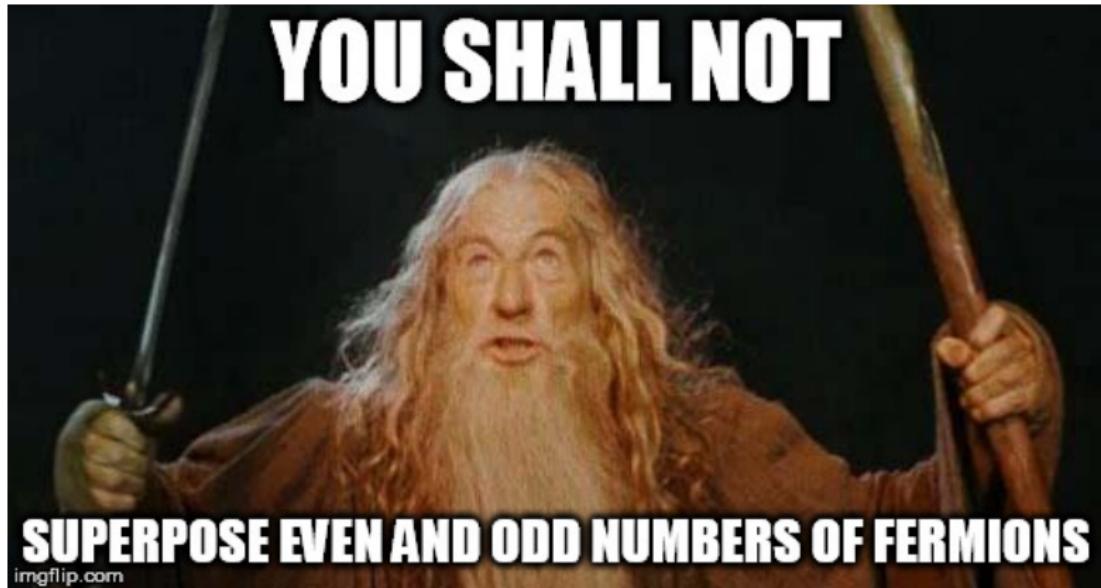
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The Parity Superselection Rule



Parity superselection prevents asymmetric pure states

Sketch of proof²

set of n fermionic modes $N = \{1, 2, \dots, n\}$ & pure state $|\psi_{\text{even}}^N\rangle$

bipartition: $M = \{\mu_i \in N | i = 1, \dots, m < n\}$ and $M^C = N/M$
write state as (w.l.o.g. m is here odd)

$$\begin{aligned} |\psi_{\text{even}}^N\rangle &= \gamma_0 |0\rangle + \sum_{i=1}^m \gamma_{\mu_i} |1_{\mu_i}\rangle |\psi_{\mu_i, \text{odd}}^{M^C}\rangle \\ &+ \sum_{\substack{i,j=1 \\ j>i}}^m \gamma_{\mu_i \mu_j} |1_{\mu_i}\rangle |1_{\mu_j}\rangle |\psi_{\mu_i \mu_j, \text{even}}^{M^C}\rangle + \dots \\ &+ \gamma_{\mu_1 \dots \mu_m} |1_{\mu_1}\rangle \dots |1_{\mu_m}\rangle |\psi_{\mu_1 \dots \mu_m, \text{odd}}^{M^C}\rangle \end{aligned}$$

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all terms have different parities

so $|\psi_{\text{even}}^N\rangle$ has different parities

but $|\psi_{\text{even}}^N\rangle$ is a pure state, so it must have the same parity

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partial trace over M^C

remove projectors on $|\psi_{\mu_1 \dots \mu_m}^{M^C}\rangle$

so the remaining terms are symmetric under exchange of μ_i and μ_j

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Origin of the parity superselection rule

But what is the reason for the SSR?

- "I don't like asymmetric pure state marginals."

Well, sure. If you are happy with this explanation.

- Charge superselection

Practical problem (?) + \exists neutral fermions

⁴ Wick, Wightman, Wigner E P, Phys. Rev. **88**, 101 (1952).

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Neutral fermions do not commute

Neutral fermions do not have a well-defined charge operator

Neutral fermions do not have a well-defined position operator

Neutral fermions do not have a well-defined momentum operator

Neutral fermions

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- Time-reversal symmetry⁴

Time-reversal symmetry implies that the charge conjugation operator commutes with the time-reversal operator.

Neutral fermions

Neutral fermions are charge conjugate to themselves under time-reversal.

Parity

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- Invariance under rotations by 2π (see ⁵)

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Quantum information in fermionic Fock space

- Density operators well defined
- Reduced states well defined (consistency conditions + SSR!)
- Entanglement \Leftrightarrow mixedness of subsystems
- Entanglement of formation well defined (computable? ⁴)

⁴ Caban, Podlaski, Rembieliński, Smoliński, Walczak, J. Phys. A: Math. Gen. 38, L79 (2005).

⁵ D'Ariano, Manessi, Perinotti, Tosini, Europhys. Lett. 107, 20009 (2014).

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Entanglement of formation is a measure of entanglement that is additive under tensor products.

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- SSR \Rightarrow no single-mode basis change, test entanglement? ⁶

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Implications for fermions in quantum information

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⁵ D'Ariano, Manessi, Perinotti, Tosini, Europhys. Lett. **107**, 20009 (2014).

⁶ Dasenbrook, Bowles, Bohr Brask, Hofer, Flindt, Brunner, New J. Phys. **18**, 043036 (2016).
Friis, New J. Phys. **18**, 061001 (2016).

Quantum information in fermionic Fock space

- Density operators well defined
- Reduced states well defined (consistency conditions + SSR!)
- Entanglement \Leftrightarrow mixedness of subsystems
- Entanglement of formation well defined (computable? ⁴)

Practical point of view

- Computable criteria, e.g., PPT, rely on tensor product
- Certain situations allow mappings to qubits
- Strange consequences: mixed max. entangled states ⁵
- SSR \Rightarrow no single-mode basis change, test entanglement? ⁶

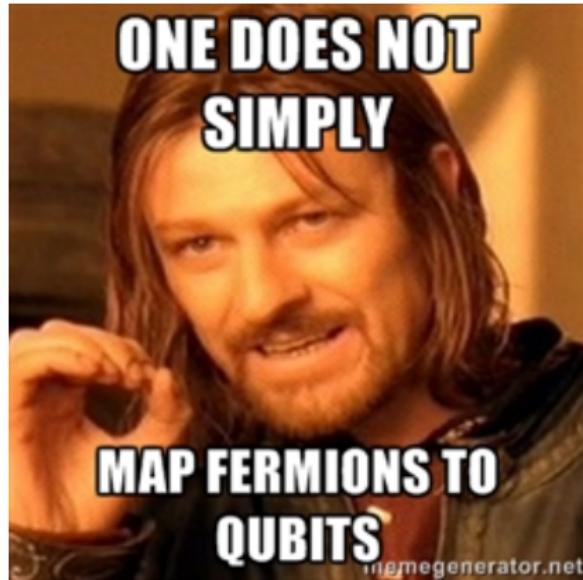
⁴ Caban, Podlaski, Rembieliński, Smoliński, Walczak, J. Phys. A: Math. Gen. **38**, L79 (2005).

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Conclusion



Thank you for your attention.

Conclusion



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