

# Heisenberg scaling in relativistic quantum metrology

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Nicolai Friis

Institute for Theoretical Physics, University of Innsbruck

in collaboration with

*Michalis Skotiniotis, Ivette Fuentes, and Wolfgang Dür*



# Motivation & Outline

## Motivation

- Quantum metrology: high-precision parameter estimation
  - quantum strategies  $\Rightarrow$  scaling gap in resources w.r.t. best classical strategy  $\Rightarrow$  Heisenberg scaling
- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

## Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results <sup>1</sup>: pure states, tracing losses & optimal scaling

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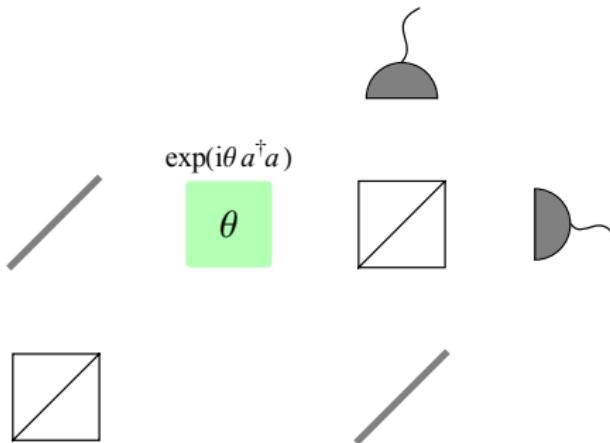
$$\exp(i\theta a^\dagger a)$$

$\theta$

single photon probe, count output photons  $N_C = c^\dagger c$   
repeat  $n$  times:  $\Delta N_C \rightarrow \Delta\theta \propto 1/\sqrt{n}$  "shot noise"

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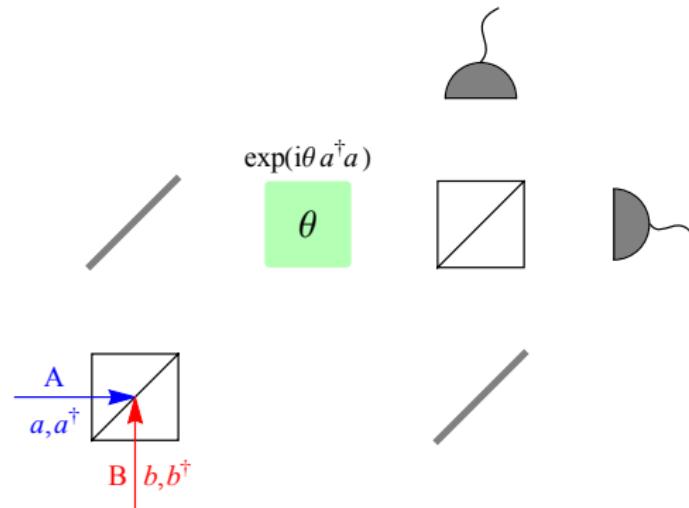
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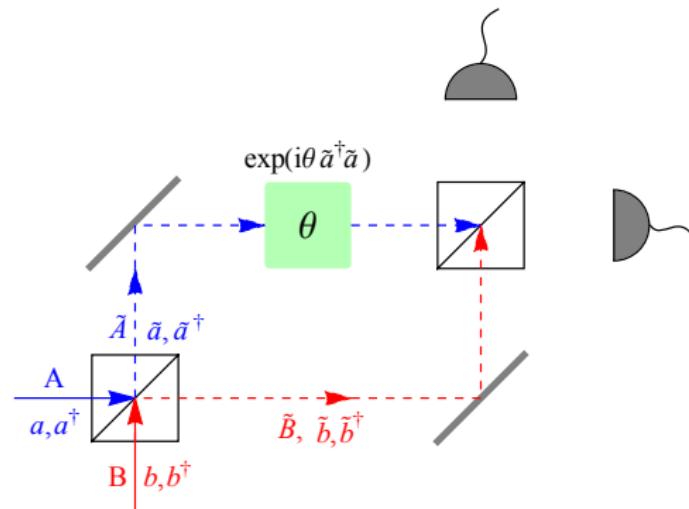
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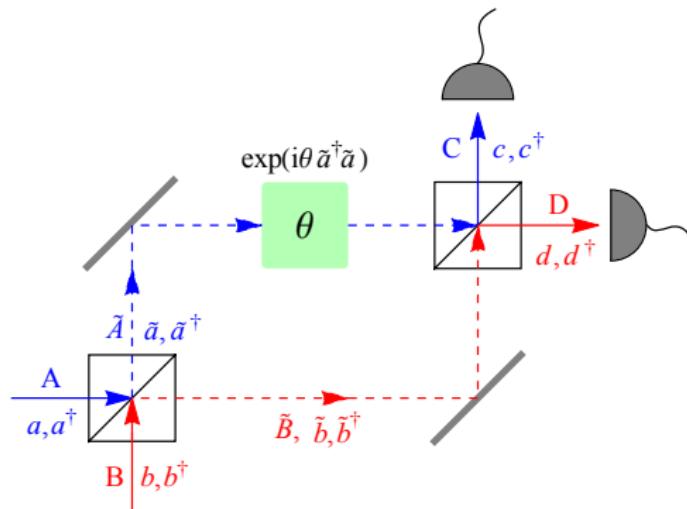


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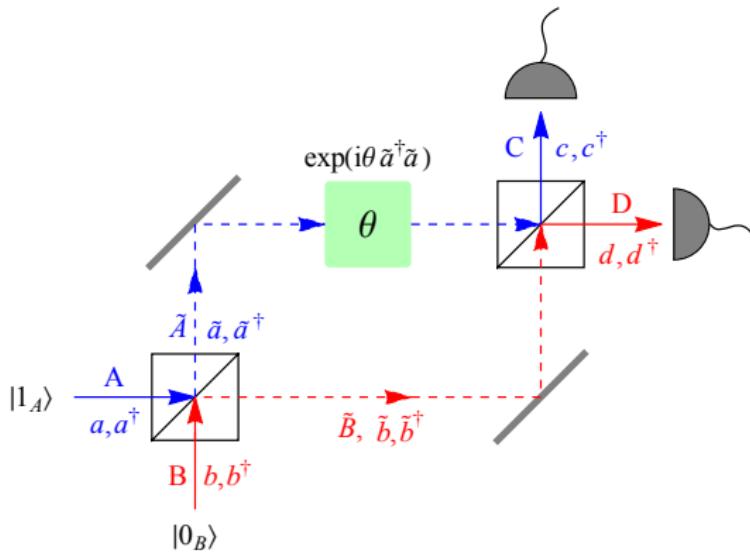


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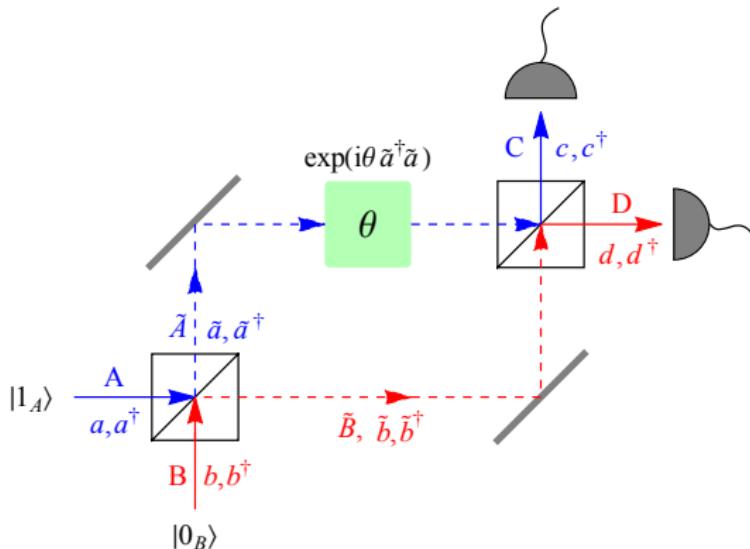


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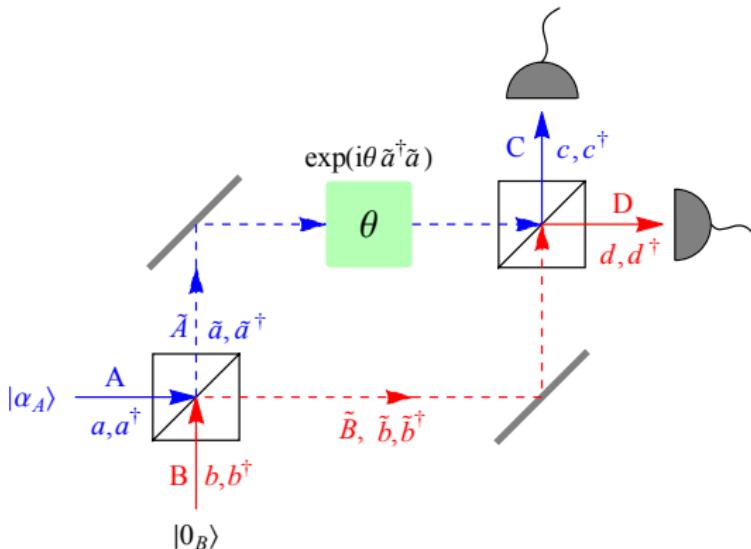
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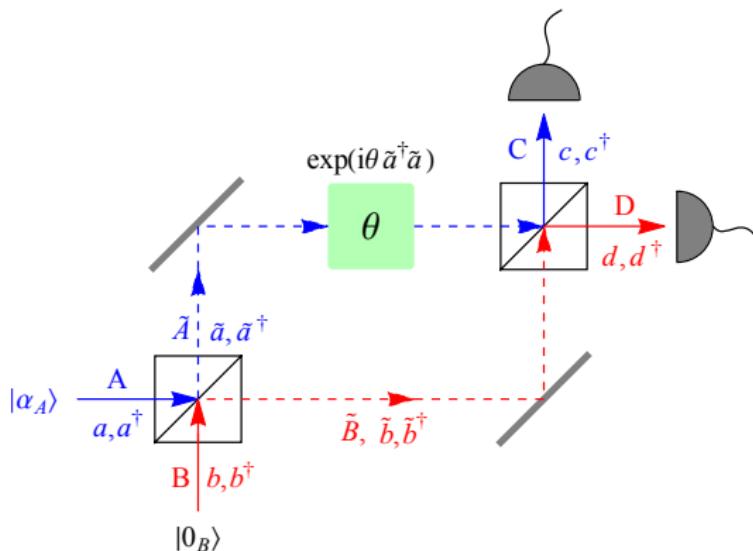


Alternatively: coherent state  $|\alpha_A\rangle$  with  $|\alpha|^2 = n$

all  $n$  photons at once, but still:  $\Delta\theta \propto 1/\sqrt{n}$  "shot noise"

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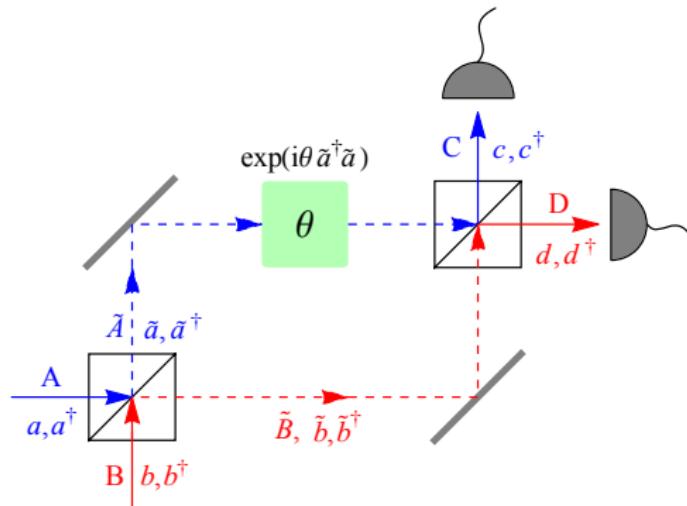
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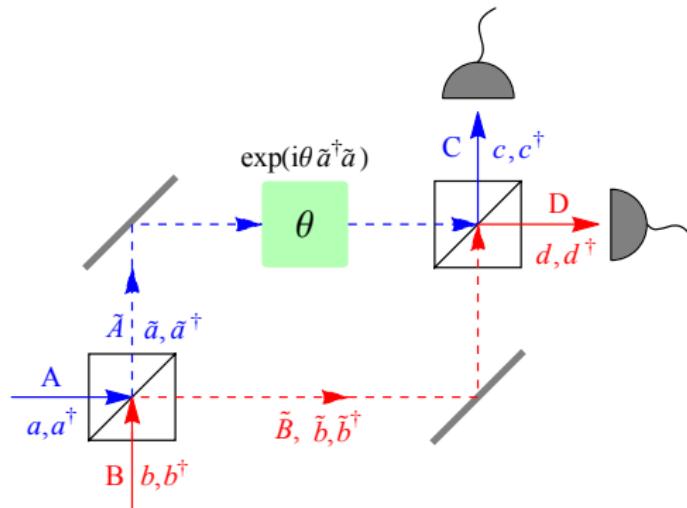


$$|\psi\rangle_{AB} = (|n+1_A\rangle|n_B\rangle + |n_A\rangle|n+1_B\rangle)/\sqrt{2}$$

Entangled input state  $|\psi\rangle_{AB}$ , average photon number  $n$   
 quadratic improvement:  $\Delta\theta \propto 1/n$  Heisenberg scaling

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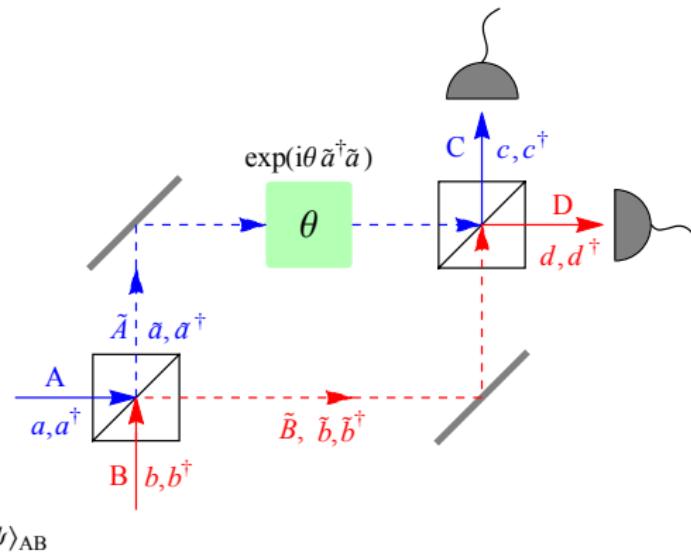


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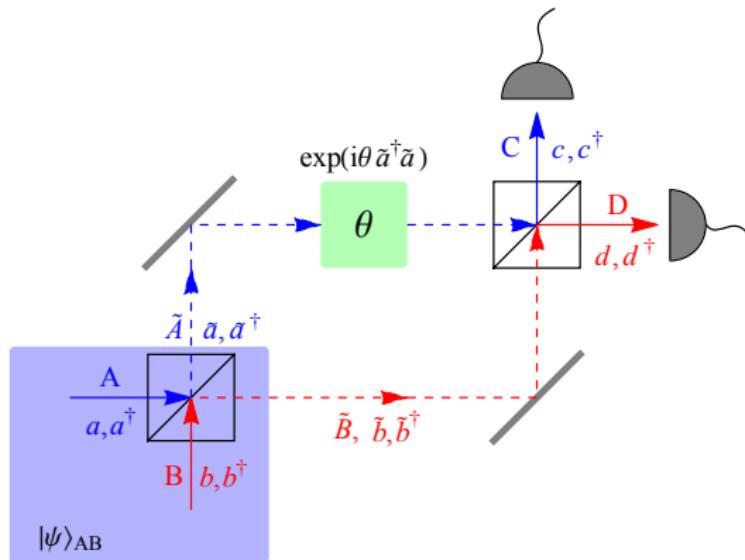
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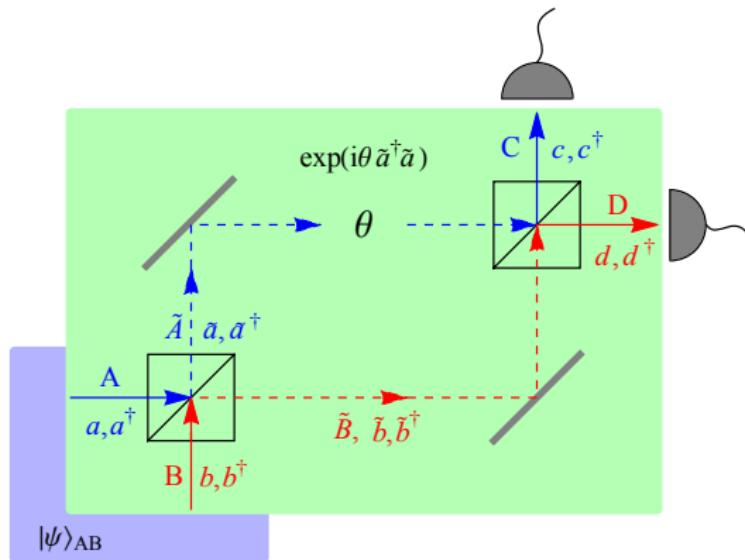
Stages: Preparation, Transformation, Measurement

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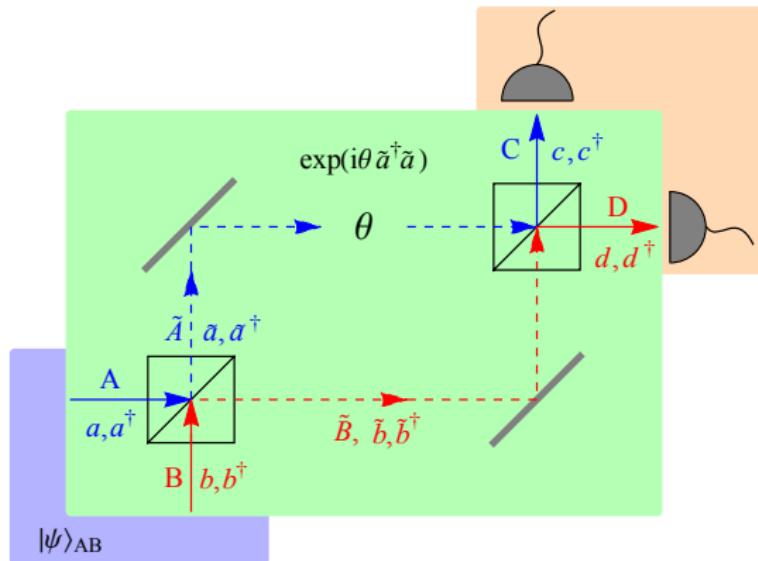
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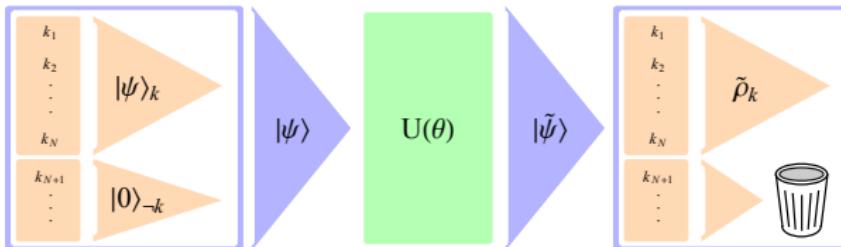
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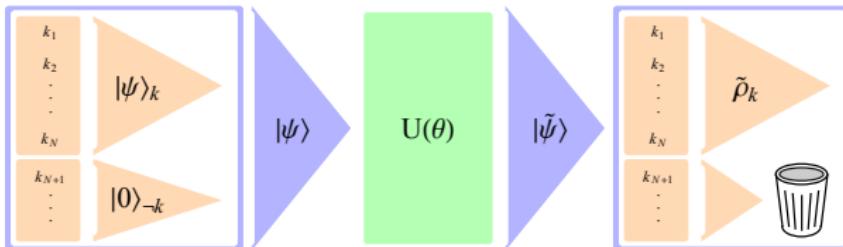
**Preparation:** only  $N$  modes controlled:  $|\psi\rangle = |\psi\rangle_k |0\rangle_{\neg k}$

**Transformation:** (Gaussian) unitary on arbitrary many modes

**Measurement:** trace out  $\neg k$ ,  $\nu$  repetitions

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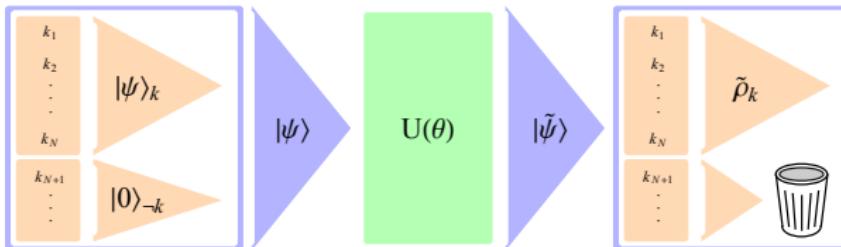
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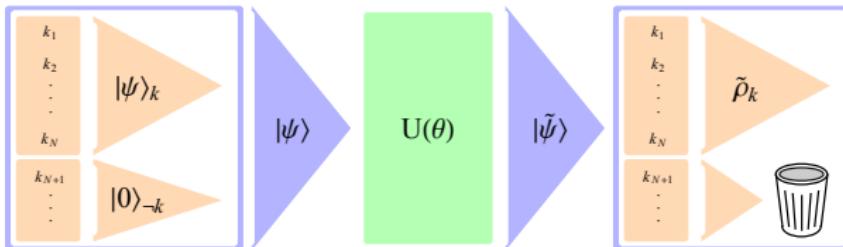
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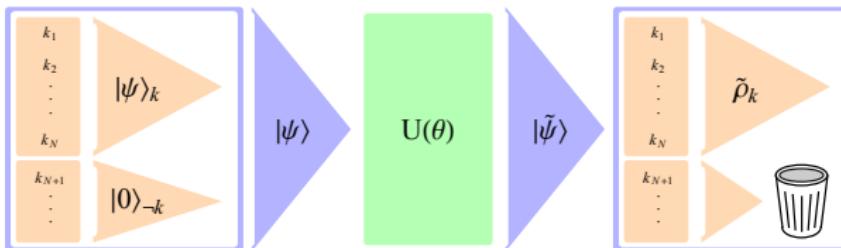
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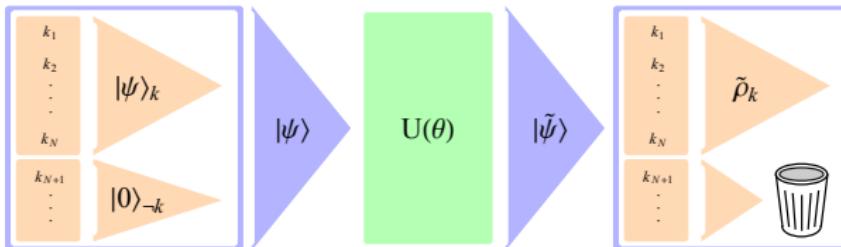
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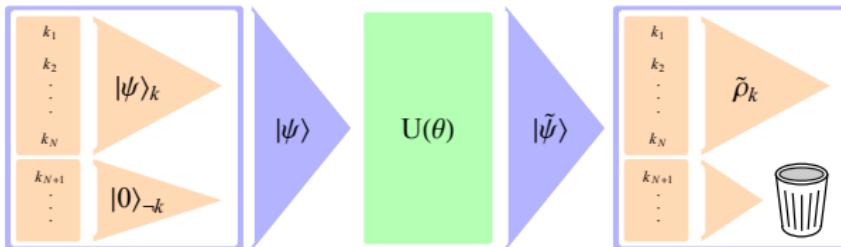
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## Perturbative approach

Unitary transformation:  $U(\theta) = U^{(0)} + \theta U^{(1)} + \theta^2 U^{(2)} + \mathcal{O}(\theta^3)$

## Total state

Transformed state:  $|\tilde{\psi}\rangle = |\tilde{\psi}^{(0)}\rangle + \theta |\tilde{\psi}^{(1)}\rangle + \theta^2 |\tilde{\psi}^{(2)}\rangle + \mathcal{O}(\theta^3)$

QFI:  $\mathcal{I}(|\tilde{\psi}\rangle) = 4 \left( \langle \tilde{\psi}^{(1)} | \tilde{\psi}^{(1)} \rangle - |\langle \tilde{\psi}^{(0)} | \tilde{\psi}^{(1)} \rangle|^2 \right) + \mathcal{O}(\theta)$

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Transformed state:  $|\tilde{\psi}\rangle = |\tilde{\psi}^{(0)}\rangle + \theta |\tilde{\psi}^{(1)}\rangle + \theta^2 |\tilde{\psi}^{(2)}\rangle + \mathcal{O}(\theta^3)$

QFI:  $\mathcal{I}(|\tilde{\psi}\rangle) = 4 \left( \langle \tilde{\psi}^{(1)} | \tilde{\psi}^{(1)} \rangle - |\langle \tilde{\psi}^{(0)} | \tilde{\psi}^{(1)} \rangle|^2 \right) + \mathcal{O}(\theta)$

## Reduced state

after tracing:  $\tilde{\rho}_k(\theta) = \tilde{\rho}_k^{(0)} + \theta \tilde{\rho}_k^{(1)} + \theta^2 \tilde{\rho}_k^{(2)} + \mathcal{O}(\theta^3)$

Reduced state QFI:

$\mathcal{I}(\tilde{\rho}_k(\theta)) = \mathcal{I}(|\tilde{\psi}\rangle) - 4 \sum_{i \neq 0} |_k \langle \tilde{\psi}^{(0)} | \neg_k \langle i | U^{(1)} | \psi \rangle_k | 0 \rangle_{\neg k}|^2 + \mathcal{O}(\theta)$

# Specialize to Gaussian Transformations

## Perturbative Bogoliubov transformation

Coefficients:  $\alpha_{mn} = \alpha_{mn}^{(0)} + \alpha_{mn}^{(1)} \theta + \alpha_{mn}^{(2)} \theta^2 + O(\theta^3)$

$$\beta_{mn} = \beta_{mn}^{(1)} \theta + \beta_{mn}^{(2)} \theta^2 + O(\theta^3)$$

Phases (free time evolution):  $\alpha_{mn}^{(0)} = \delta_{mn} G_n = \delta_{mn} \exp(i\phi_n)$

Vacuum:  $|0\rangle \mapsto |0\rangle - \theta \frac{1}{2} \sum_{p,q} G_q^* \beta_{pq}^{(1)*} a_p^\dagger a_q^\dagger |0\rangle + \mathcal{O}(\theta^2)$

## Tracing loss

QFI:  $\mathcal{I}(\tilde{\rho}_k(\theta)) = \mathcal{I}(|\tilde{\psi}\rangle) - \Delta_{\text{tr}}(|\psi\rangle_k) + \mathcal{O}(\theta)$

Tracing loss:  $\Delta_{\text{tr}}(|\psi\rangle_k) \geq \Delta_{\text{tr}}(|0\rangle_k) = 2 \sum_{p,q \notin k} |\beta_{pq}^{(1)}|^2$

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# Optimal Scaling

- Have QFI for any initial state
- Know how to pick states with minimal tracing loss

$$\text{e.g., } \mathcal{I}(|n_k\rangle) = 2n(n+1)|\beta_{kk}^{(1)}|^2 + 4n \sum_{p \neq k} (|\alpha_{pk}^{(1)}|^2 + |\beta_{pk}^{(1)}|^2) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

or for  $\beta_{kk}^{(1)} = \beta_{k'k'}^{(1)} = 0$

$$\begin{aligned} \mathcal{I}(|n_k\rangle|m_{k'}\rangle) = & 8mn(|\alpha_{kk'}^{(1)}|^2 + |\beta_{kk'}^{(1)}|^2) + 4n \sum_{p \neq k} (|\alpha_{pk}^{(1)}|^2 + |\beta_{pk}^{(1)}|^2) \\ & + 4m \sum_{p \neq k'} (|\alpha_{pk'}^{(1)}|^2 + |\beta_{pk'}^{(1)}|^2) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta) \end{aligned}$$

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OPTIMAL

# Summary (for proofs please see <sup>1</sup>)

- Parameter estimation framework for **perturbative Bogoliubov transformations**
  - Works for **any** initial states (not just Gaussian as in <sup>3</sup>)
    - Quantify & minimize tracing losses
    - Show that Heisenberg scaling is optimal
  - Determine family of **ALL** optimally scaling & minimal tracing loss states

<sup>1</sup> N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, e-print arXiv:1502.07654 [quant-ph] (2015).

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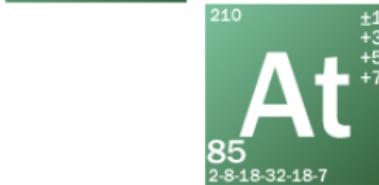
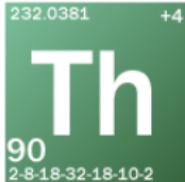
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Thank you for your  
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