GAUSSIAN OPERATIONS FOR WORK EXTRACTION AND STORAGE

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Quantum Thermodynamics

Thermodynamics in the quantum regime, here: as a resource theory

- Resource: Work/Energy
- Free states: thermal states $\tau(\beta) = \frac{e^{-\beta H}}{\mathcal{Z}}$
- Free operations: energy conserving unitaries

➡ Interested in extracting, distributing & storing energy

➡ What are the fundamental limitations?

➡ What can be achieved practically? e.g., with Gaussian operations

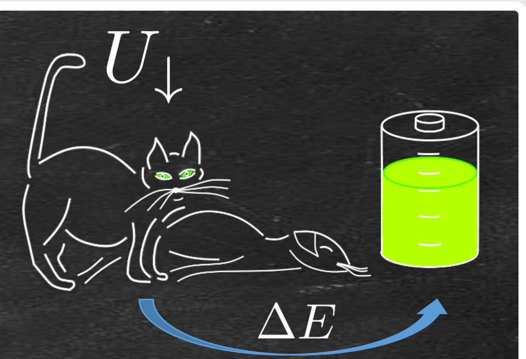
Work extraction

How can work be extracted from a (quantum) system?

➡ Standard paradigm: Unitary $U_\downarrow$ on qantum system to lower energy

➡ Store energy ΔE in battery to conserve energy

➡ Quantum states useful only if energy can be lowered by unitaries



Otherwise states are called *passive* , e.g., thermal state $\tau(\beta) = \frac{e^{-\beta H}}{\mathcal{Z}}$

On the other hand, two thermal states at different temperatures $\tau(\beta)\otimes\tau(\beta')$ are not passive

But, how complicated are unitaries for arbitrary states? Can such unitaries be realized in practice?

If not, how much energy may be extracted with practical operations?

Gaussian Unitaries

- Class of easily* implementable operations (*generated by Hamiltonians at most quadratic in mode operators ➡ lowest in hierarchy of operations, require only two-body interactions)
- Map all Gaussian states to Gaussian states

Gaussian states:
- quantum states of bosonic modes with Gaussian Wigner function
- Fully described by vector of first moments $\overline{\mathbb{X}} = \langle\,\mathbb{X}\,\rangle = (\langle\,\mathbb{X}_i\,\rangle)_i$ and 2[nd] moments collected in covariance matrix Γ with components $\Gamma_{ij} = \langle\,\mathbb{X}_i\mathbb{X}_j + \mathbb{X}_j\mathbb{X}_i\,\rangle - 2\langle\,\mathbb{X}_i\,\rangle\langle\,\mathbb{X}_j\,\rangle$
- Quadrature operators, e.g., for n[th] mode $\mathbb{X}_{2n-1} = (a_n + a_n^\dagger)/\sqrt{2}$, $\mathbb{X}_{2n} = -i(a_n - a_n^\dagger)/\sqrt{2}$
- Symplectic form $\Omega_{mn} = i[\mathbb{X}_m, \mathbb{X}_n]$

➡ Gaussian unitaries: affine symplectic maps $(S,\xi):\mathbb{X}\mapsto S\mathbb{X}+\xi$ displacements $D(\xi) = \exp(i\mathbb{X}^T\Omega\xi)$ symplectic $S\,\Omega\,S^T = \Omega$

Gaussian Passivity

Definition: Any (not necessarily Gaussian) state is called *Gaussian-passive* if its average energy cannot be reduced by Gaussian unitaries.

Theorem 1: Any (not necessarily Gaussian) state of two (noninteracting) bosonic modes with frequencies ω_a and $\omega_b \geq \omega_a$ is Gaussian-passive if and only if its first moments vanish, $\langle\,\mathbb{X}\,\rangle = 0$, and its covariance matrix Γ is either
(i) in Williamson normal form $\Gamma = \mathrm{diag}\{\nu_a,\nu_a,\nu_b,\nu_b\}$, with $\nu_a \geq \nu_b$ for $\omega_a < \omega_b$. Or, in the case where $\omega_a = \omega_b$
(ii) in standard form $\Gamma = \begin{pmatrix} a\mathbb{1} & C\\ C & b\mathbb{1}\end{pmatrix}$, with $C = c\mathbb{1}$.

Corollary: Arbitrary state of n bosonic modes Gaussian-passive iff all two-mode marginals are Gaussian-passive.

Passivity vs. Gaussian Passivity

Passivity $\Longrightarrow$ Gaussian passivity But Gaussian passivity $\not\Longrightarrow$ passivity

For Gaussian initial states: Gaussian passivity $\Longrightarrow$ passivity

For general initial state: ➡ *Gaussian ergotropy* (energy extractable with Gaussian unitaries) uniquely defined by $\overline{\mathbb{X}} = \langle\,\mathbb{X}\,\rangle$ and Γ

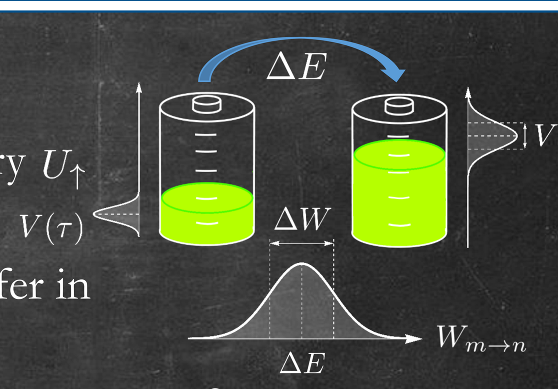
➡ Corresponding Gaussian-passive state not unique

Theorem 2: The 1[st] and 2[nd] moments of any Gaussian-passive state with entropy S_0 are compatible with a (non-Gaussian) state ρ with $S(\rho) = S_0$ and for which the maximal amount of energy (the energy difference to a thermal state with entropy S_0) is extractable by general unitaries.

➡ Gap between Gaussian passivity and passivity is maximal

Work Storage

Task: Transfer energy ΔE to empty* battery via unitary $U_\uparrow$ [*no extractable work ➡ initially thermal $\tau(\beta)$]



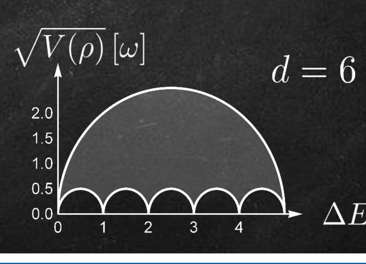
Unitaries $U_\uparrow : \tau\mapsto\rho$ with $E(\rho) > E(\tau)$ exist but differ in properties of $U_\uparrow$ and ρ , e.g.

- precision (energy variance) $V(\rho) = (\Delta H_\rho)^2 = \langle\,H^2\,\rangle_\rho - \langle\,H\,\rangle_\rho^2$
- Work fluctuations $(\Delta W)^2 = \sum_{m,n} p_{m\to n}(W_{m\to n} - \Delta E)^2$ with $W_{m\to n} = E_n - E_m$ and transition probability $p_{m\to n} = p_m\,|\langle\,n\,|\,U_\uparrow\,|\,m\,\rangle|^2$ where $p_m = \langle\,m\,|\,\tau\,|\,m\,\rangle$

Example: Hamiltonian with equal spacing $E_{n+1} - E_n = \omega\;\;\forall\, m$

For $T = 0$: $\tau = |0\rangle\langle 0|$ Worst case: $V(\rho) = \Delta E(\omega(d-1) - \Delta E)$

Best case: $V(\rho) = (\Delta E - \lfloor\Delta E\rfloor)(\lceil\Delta E\rceil - \Delta E)$



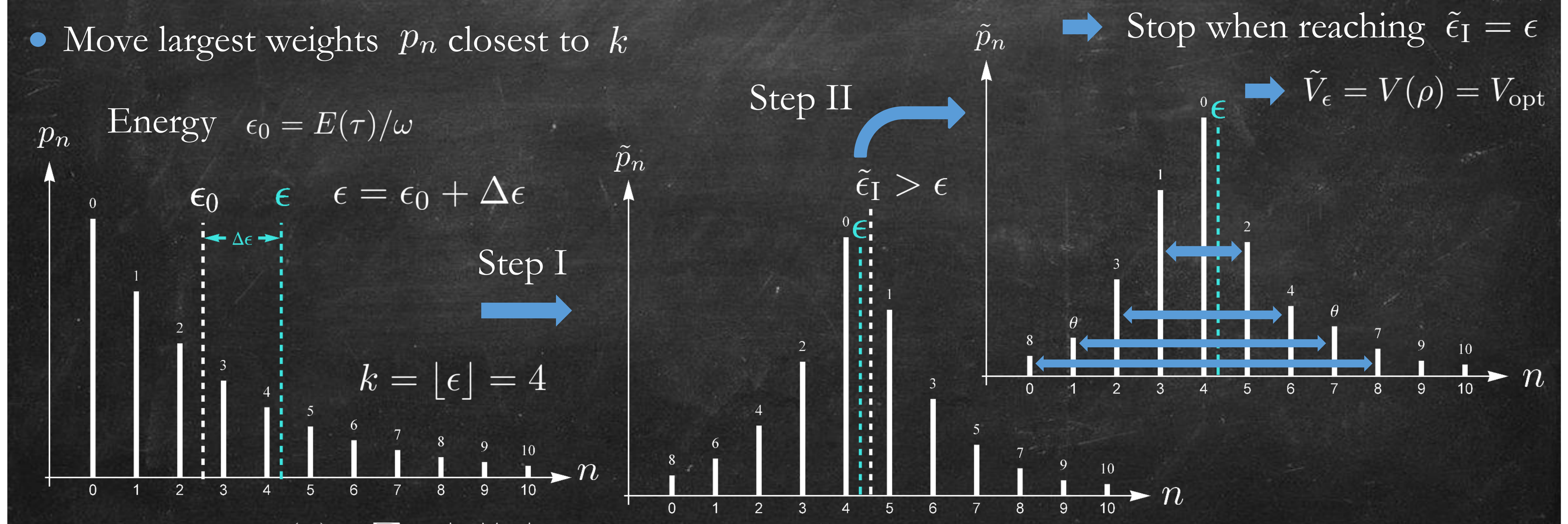
Optimal Precision Battery Charging

Step I:
- Identify level k closest to target energy ϵ
- Move largest weights p_n closest to k

Step II:
- Identify level pairs to adjust energy correctly
- Rotate between levels, start with minimal $\frac{\Delta\tilde{V}_\epsilon}{|\Delta\tilde{\epsilon}|}$

➡ Stop when reaching $\tilde{\epsilon}_{\mathrm{I}} = \epsilon$

➡ $\tilde{V}_\epsilon = V(\rho) = V_{\mathrm{opt}}$

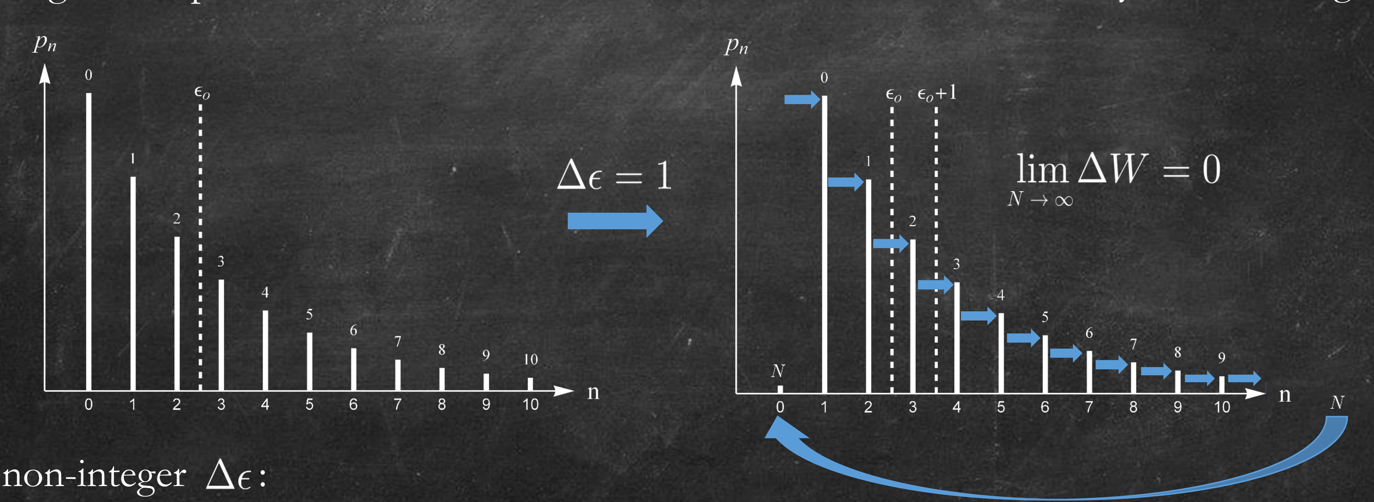


Initial state $\tau(\beta) = \sum_n p_n\,|n\rangle\langle n|$

➡ Minimal $\tilde{V}_\epsilon$ [mean square deviation from ϵ] but $\tilde{\epsilon}_{\mathrm{I}} \neq \epsilon$

Minimal Fluctuations

For integer multiples of ω: $\Delta W = 0$ When $\Delta\epsilon = m\in\mathbb{N}$ ➡ Shift by m to the right



For non-integer $\Delta\epsilon$:
- Start shifting at $k = \lceil(\beta\omega)^{-1}\ln(1/\Delta\epsilon)\rceil > 0$
- Fine-tune: rotation between $k-1$ and k ➡ $(\Delta W)^2 = (\Delta E - \lfloor\Delta E\rfloor)(\lceil\Delta E\rceil - \Delta E) = V_{\mathrm{opt}}(T=0)$

Gaussian Battery Charging

Limitation of Gaussian Unitaries?

Phase space description: Wigner representation $\rho\mapsto\mathcal{W}(x,p) = \frac{1}{(2\pi)^N}\int\! dy\, e^{-ipy}\langle x+\frac{y}{2}|\rho|x-\frac{y}{2}\rangle$

Observables: $\langle\,\hat{G}\,\rangle_\rho = \mathrm{Tr}(\hat{G}\rho) = \int\! dxdp\,\mathcal{W}(x,p)\,g(x,p)$ with $g(x,p) = \int\! dy\, e^{ipy}\langle x-\frac{y}{2}|\hat{G}|x+\frac{y}{2}\rangle$

Gaussian states $\mathcal{W}(\xi) = \frac{1}{\pi^N\sqrt{\det(\Gamma)}}\exp[-(\xi-\overline{\mathbb{X}})^T\Gamma^{-1}(\xi-\overline{\mathbb{X}})]$ $\overline{\mathbb{X}} = \langle\mathbb{X}\rangle_\rho,\;\xi = (x_1,p_1,\ldots,x_N,p_N)^T$

Energy: $\frac{E(\rho)}{\omega} = \frac{1}{4}[\mathrm{Tr}(\Gamma) - 2] + \frac{1}{2}\|\overline{\mathbb{X}}\|^2$ Variance: $(\frac{\Delta H}{\omega})^2 = \frac{1}{2}\overline{\mathbb{X}}^T\Gamma\,\overline{\mathbb{X}} + \frac{1}{8}[\mathrm{Tr}(\Gamma^2) - 2]$

Precision: example: pure displacement $D(\alpha)$

$\frac{\Delta E}{\omega} = \frac{1}{2}\|\overline{\mathbb{X}}\|^2 = \frac{1}{2}|\alpha|^2$

$(\frac{\Delta H}{\omega})^2 = \frac{1}{2}\coth(\frac{\beta\omega}{2})\|\overline{\mathbb{X}}\|^2 + \frac{V(\tau)}{\omega^2}$

as $\boxed{\Delta E\to\infty : V(\rho)/\Delta E\to\mathrm{const.}}$

For general Gaussian unitaries

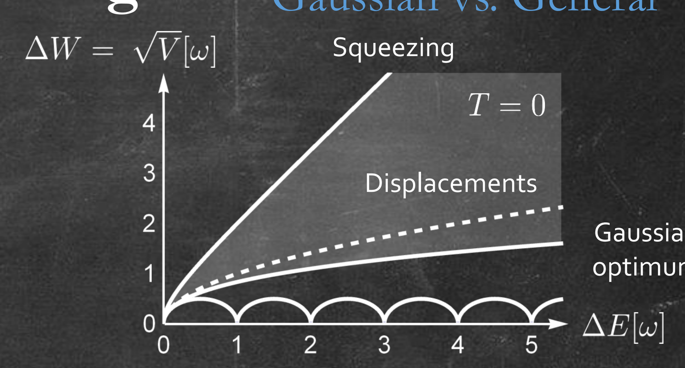
- **Optimal**: combination of squeezing & displacement as $\boxed{\Delta E\to\infty : \;V(\rho)/\Delta E\to 0}$
- **Worst case:** pure single-mode squeezing

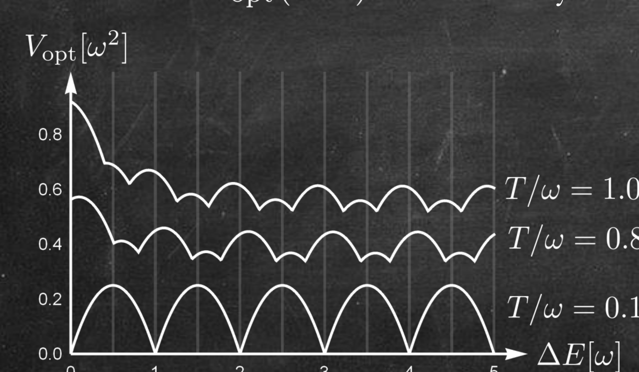
Fluctuations:
- **Optimal**: combination of squeezing & displacement: as $\Delta E\to\infty$: $\frac{(\Delta W)^2}{\Delta E}\to 0$
- **Worst case:** in general also combination of squeezing & displacement

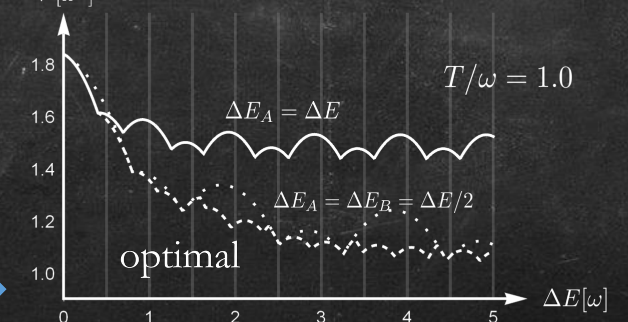
Precision: Optimal and Worst Strategies

Single-mode batteries
- For $T > 0$: variance may decrease
- For fixed T: $V_{\mathrm{opt}}(\Delta E)$ bounded by constants

Gaussian vs. General







Multi-mode batteries
- Already local unitaries provide advantage ➡
- Correlations can occur during step II ➡ Correlations can help but play no central role

References: ● E. G. Brown, N. Friis, and M. Huber, New J. Phys. **18**, 113028 (2016) [arXiv:1608.04977] ● N. Friis and M. Huber, Quantum **2**, 61 (2018) [arXiv:1708.00749]

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