Gaussian Operations for Work Extraction and Storage

+ some remarks about the energy cost of measurements

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Motivation and Introduction

Quantum Thermodynamics ➡ thermodynamics in the quantum regime

- Thermodynamic laws in the quantum domain
- Equilibration & thermalization of quantum systems
- Quantum Thermodynamics vs Statistical Mechanics
- (Autonomous) Quantum heat engines

Recent review from QI perspective: Goold, Huber, Riera, del Rio, P. Skrzypczyk, *J. Phys. A: Math. Theor.* ***49***, *143001 (2016)* [arXiv:1505.07835].

e.g., fluctuation-dissipation theorems: Crooks [*Phys. Rev. E* ***60*** *2721 (1999)*], Tasaki [*arXiv: cond-mat/0009244*], Jarzynski [*Phys. Rev. Lett.* ***78*** *2690 (1997)*], Talkner, Lutz, Hänggi [Phys. Rev. E 75, 050102(R) (2007)]

Here: Quantum Thermodynamics as a resource theory

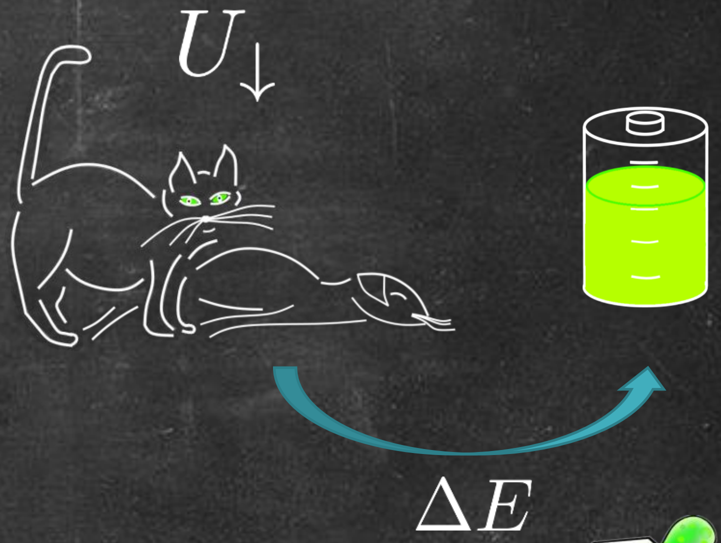
- Resource: Work/Energy
- Free states: thermal states $\tau(\beta) = \frac{e^{-\beta H}}{\mathcal{Z}}$
- Free operations: energy conserving unitaries

➡ Interested in extracting, distributing & storing energy (fundamental limitations?)

➡ What can be achieved practically? ➡ e.g., with Gaussian operations

Work extraction

How can work be extracted from a (quantum) system?



Standard paradigm: Unitary on qantum system to lower energy

➡ Store energy in battery to conserve energy

➡ Quantum states useful only if energy CAN be lowered by unitaries

➡ Otherwise states are called *passive*, e.g., thermal state $\tau(\beta) = \frac{e^{-\beta H}}{\mathcal{Z}}$

On the other hand, two thermal states at different temperatures $\tau(\beta) \otimes \tau(\beta')$ ➡ not passive

BUT: How complicated are unitaries for arbitrary states? Can such unitaries be realized in practice?

➡ If not, how much energy may be extracted with practical operations?

Pusz & Woronowicz,
Commun. Math. Phys. 58, 273 (1978)

Gaussian passivity

Class of practically implementable operations:

(operations that map Gaussian states to Gaussian states)

Gaussian unitaries

Recall: Gaussian states fully described by 1st moments $\langle\, \mathbb{X}_i \,\rangle$ and 2nd moments Γ

i.e., covariance matrix $\Gamma_{ij} = \langle\, \mathbb{X}_i\mathbb{X}_j + \mathbb{X}_j\mathbb{X}_i \,\rangle - 2\langle\, \mathbb{X}_i \,\rangle\langle\, \mathbb{X}_j \,\rangle$ with

quadrature operators $\mathbb{X}_{2n-1} = (a_n + a_n^\dagger)/\sqrt{2}$ and $\mathbb{X}_{2n} = -i(a_n - a_n^\dagger)/\sqrt{2}$

Definition: Any (not necessarily Gaussian) state is called *Gaussian-passive* if its average energy cannot be reduced by Gaussian unitaries.

Phase space displacements $D(\xi) = \exp(i\mathbb{X}^T\Omega\xi)$

Gaussian unitaries: affine maps $(S,\xi) : \mathbb{X} \mapsto S\mathbb{X} + \xi$

Symplectic transformations $S\,\Omega\,S^T = \Omega$ with $\Omega_{mn} = i\,[\mathbb{X}_m\,, \mathbb{X}_n\,]$

Theorem (Gaussian passive states)

Any (not necessarily Gaussian) state of two (noninteracting) bosonic modes with frequencies ω_a and $\omega_b \geq \omega_a$ is Gaussian-passive if and only if its first moments vanish, $\langle \, \mathbb{X} \, \rangle = 0$, and its covariance matrix Γ is either

(i) in Williamson normal form $\Gamma = \mathrm{diag}\{\nu_a, \nu_a, \nu_b, \nu_b\}$, with $\nu_a \geq \nu_b$ for $\omega_a < \omega_b$. Or, in the case where $\omega_a = \omega_b$,

(ii) in standard form $\Gamma = \begin{pmatrix} a\mathbb{1} & C \\ C & b\mathbb{1} \end{pmatrix}$, with $C = c\mathbb{1}$.

Sketch of Proof:

1) Start with most general combination of 1st and 2nd moments.

2) Successively apply Gaussian unitaries to reduce average energy as much as possible.

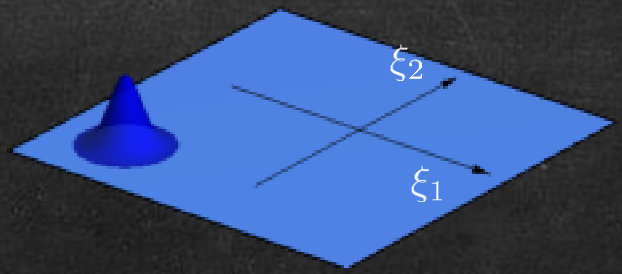
3) Show that the final state has the lowest energy in any Gaussian unitary orbit.

First note: average energy for single mode

$$E(\rho) = \omega\,\mathrm{Tr}(\rho\, a^\dagger a) = \omega\left(\tfrac{1}{4}\left[\mathrm{Tr}(\Gamma) - 2\right] + \tfrac{1}{2}\|\langle\, \mathbb{X} \rangle\|^2\right)$$

Step 1 Displacements $D(\xi = -\langle\, \mathbb{X}\,\rangle)$

Shift first moments of every mode to $\langle\, \mathbb{X}\,\rangle = 0$



Step 2 Local symplectic operations

Note: Every two-mode covariance matrix Γ can be brought to standard form by local symplectic operations $S_{\text{loc}} = S_{\text{loc},a} \oplus S_{\text{loc},b}$, i.e., $S_{\text{loc}}\,\Gamma\,S_{\text{loc}}^T = \Gamma_{\text{st}} = \begin{pmatrix} a\,\mathbb{1} & C \\ C & b\,\mathbb{1} \end{pmatrix}$, with $C = \text{diag}\{c_1, c_2\}$.

Loc. sympl. ransformations decompose as $S_{\text{loc},i} = R(\theta_i)\,S(r_i)\,R(\phi_i)$

$$S(r_i) = \begin{pmatrix} e^{-r_i} & 0 \\ 0 & e^{r_i} \end{pmatrix}$$

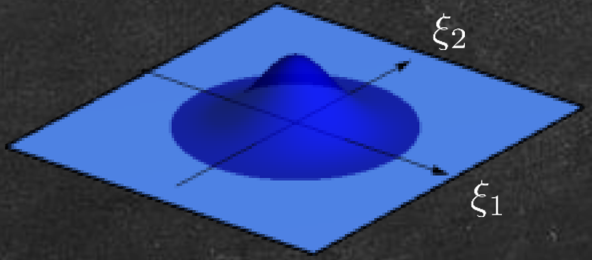
$$R(\theta_i) = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}$$

rotations

single-mode squeezing

conversely $\Gamma = (S_{\text{loc}}^{-1})\,\Gamma_{\text{st}}\,(S_{\text{loc}}^{-1})^T$

$$E(\Gamma) = \tfrac{\omega_a}{2}\left(a\cosh(2r_a) - 1\right) + \tfrac{\omega_b}{2}\left(b\cosh(2r_b) - 1\right)$$



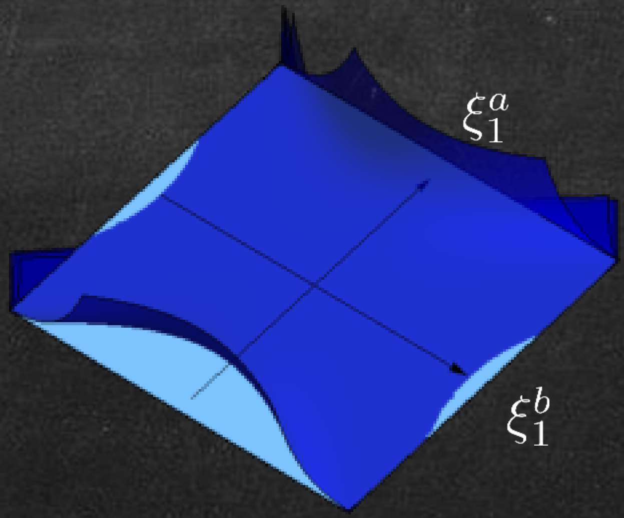
Bring Γ to standard form using local rotations and single-mode squeezing

Step 3 Two-mode squeezing

Note: after exploiting all local Gaussian unitaries we are left with $\Gamma = \begin{pmatrix} a\,\mathbb{1} & C \\ C & b\,\mathbb{1} \end{pmatrix}$, with $C = \mathrm{diag}\{c_1, c_2\}$.

But in general $c_1 \neq c_2$ ➡ Can reduce energy using two-mode squeezing (*and free local rotations)

$$S_{\mathrm{TMS}} = \begin{pmatrix} \cosh(r)\mathbb{1} & \sinh(r)\sigma_z \\ \sinh(r)\sigma_z & \cosh(r)\mathbb{1} \end{pmatrix} \quad \text{with} \quad r = -\tfrac{1}{2}\,\mathrm{artanh}\left(\tfrac{c_1 - c_2}{a+b}\right) \overset{*}{\Rightarrow} \widehat{\Gamma} = \begin{pmatrix} \tilde{a}\,\mathbb{1} & c\,\mathbb{1} \\ c\,\mathbb{1} & \tilde{b}\,\mathbb{1} \end{pmatrix}$$

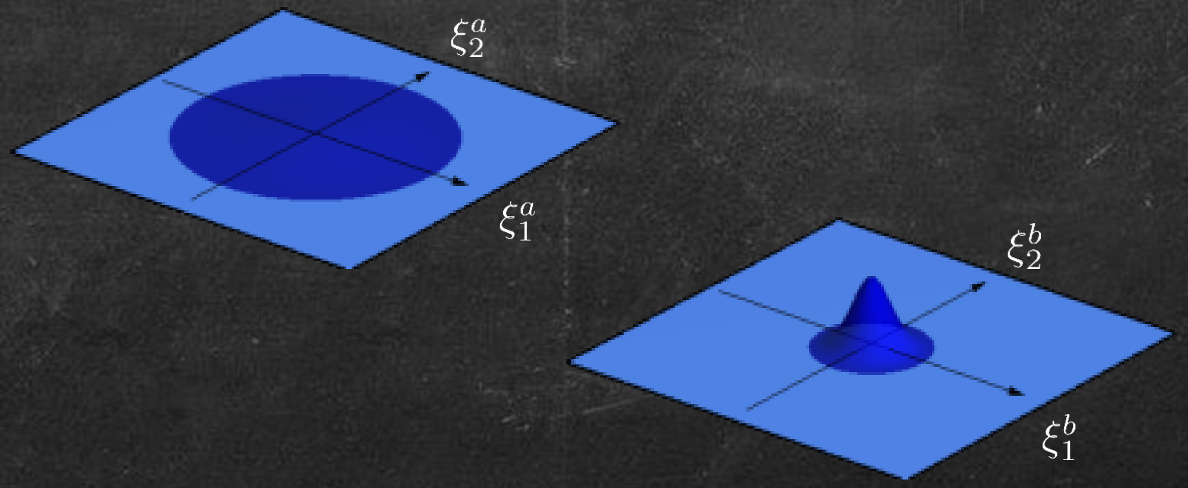


Step 4 "Beam splitting"

If $\omega_a = \omega_b$ ➡ done (beam splitting leaves excitation number invariant)

If $\omega_a < \omega_b$ ➡ can shift excitation to lower-energy mode $S_{\mathrm{BS}}(\theta) = \begin{pmatrix} \cos(\theta)\,\mathbb{1} & \sin(\theta)\,\mathbb{1} \\ \sin(\theta)\,\mathbb{1} & -\cos(\theta)\,\mathbb{1} \end{pmatrix}$

with $\theta = \begin{cases} \frac{1}{2}\arctan(\frac{2c}{a-b}) & \text{if } a \geq b \\ \frac{1}{2}\arctan(\frac{2c}{a-b}) + \frac{\pi}{2} & \text{if } a < b \end{cases}$



➡ $\Gamma = \mathrm{diag}\{\nu_a, \nu_a, \nu_b, \nu_b\}$

with $\nu_a \geq \nu_b$

Observations and Consequences

Passivity $\Longrightarrow$ Gaussian passivity but Gaussian passivity $\not\Longrightarrow$ Passivity

However, for Gaussian states: Gaussian passivity $\Longrightarrow$ passivity

Example: two-mode thermal state, different frequencies & temperatures $\tau(\omega_a, \beta_a) \otimes \tau(\omega_b, \beta_b)$

with $\tau(\omega, \beta) = (1 - e^{-\beta\omega}) \sum_n e^{-n\beta\omega} \,|\, n \,\rangle\langle\, n \,|$ ➡ $\Gamma = \mathrm{diag}\{\nu_a, \nu_a, \nu_b, \nu_b\}$ & $\nu_i = \coth(\frac{\omega_i}{2T_i})$

Gaussian-passive iff $\frac{\omega_a}{\omega_b} < \frac{T_a}{T_b}$ ➡ Same condition as for passivity for two thermal states

General initial state ➡ lowest energy achievable with Gaussian unitaries **unique**

➡ Corresponding Gaussian-passive state **not unique**

Corollary: Arbitrary state of n bosonic modes Gaussian-passive iff all two-mode marginals are Gaussian-passive

Gap between Passivity and Gaussian Passivity

After reaching Gaussian passivity: How much extractable work is potentially left?

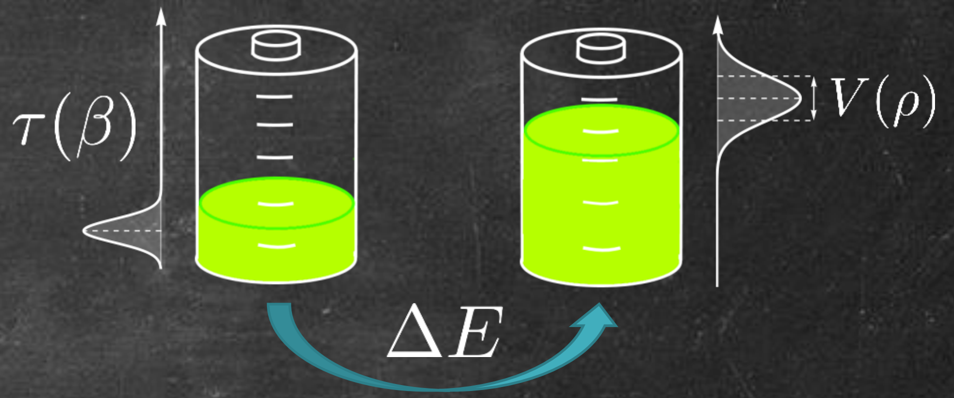
Lemma: 1st & 2nd moments of any Gaussian-passive state are compatible with a (non-Gaussian) **pure** state for which the entire energy is extractable by general unitary transformations.

Theorem: 1st & 2nd moments of any Gaussian-passive state with entropy S_o are compatible with a (non-Gaussian) **mixed** state w. same entropy for which the maximal amount of energy (the energy difference to the thermal state of entropy S_o) is extractable in principle.

For proofs please see E. G. Brown, N. Friis, and M. Huber, New J. Phys. **18**, 113028 (2016) [arXiv:1608.04977].

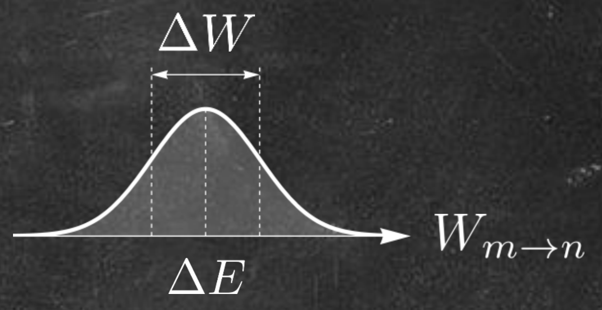
Work storage

Transfer energy ΔE to quantum battery via unitary $U_\uparrow$



Unitaries $U_\uparrow : \tau \mapsto \rho$ $\exists$ but have different properties

e.g., **variance** $V(\rho) = (\Delta H_\rho)^2 = \langle\, H^2 \,\rangle_\rho - \langle\, H \,\rangle_\rho^2$



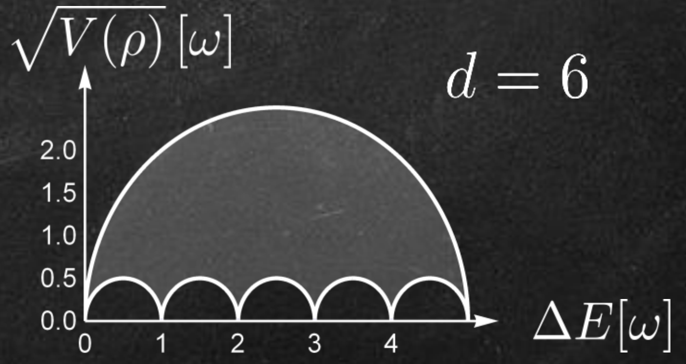
or **work fluctuations** $(\Delta W)^2 = \sum_{m,n} p_{m\to n}(W_{m\to n} - \Delta E)^2$

with $W_{m\to n} = E_n - E_m$ & $p_{m\to n} = p_m \,|\langle n|U_\uparrow|m\rangle|^2$ & $p_m = \langle m|\tau|m\rangle$

Example: equal spacing $E_{n+1} - E_n = \omega \;\; \forall\, m$ and $T = 0$

Worst case: $V(\rho) = \Delta E\big(\omega(d-1) - \Delta E\big)$

Best case: $V(\rho) = \big(\Delta E - \lfloor \Delta E \rfloor\big)\big(\lceil \Delta E \rceil - \Delta E\big)$



Optimal Precision Charging

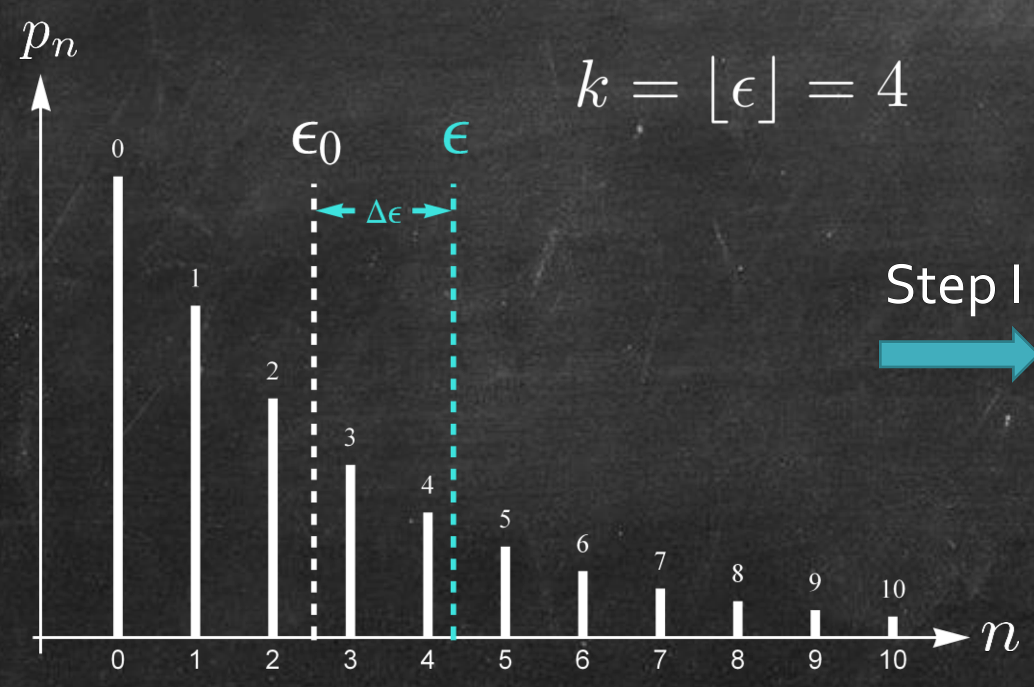
Initital state $\tau(\beta) = \sum_n p_n \, |n\rangle\langle n|$

Energy $\epsilon_0 = E(\tau)/\omega$

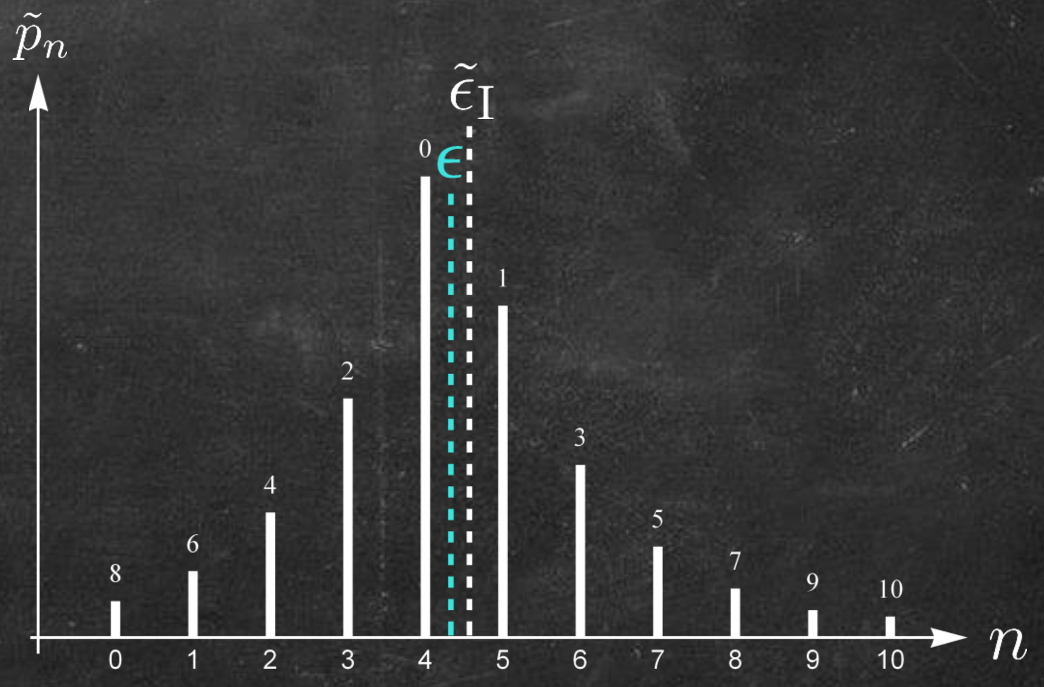
Optimal strategy has 2 Steps: Step I:

- Identify level k closest to target energy $\epsilon = \epsilon_0 + \Delta\epsilon$
- Move largest weights p_n closest to k

$k = \lfloor \epsilon \rceil = 4$



Step I



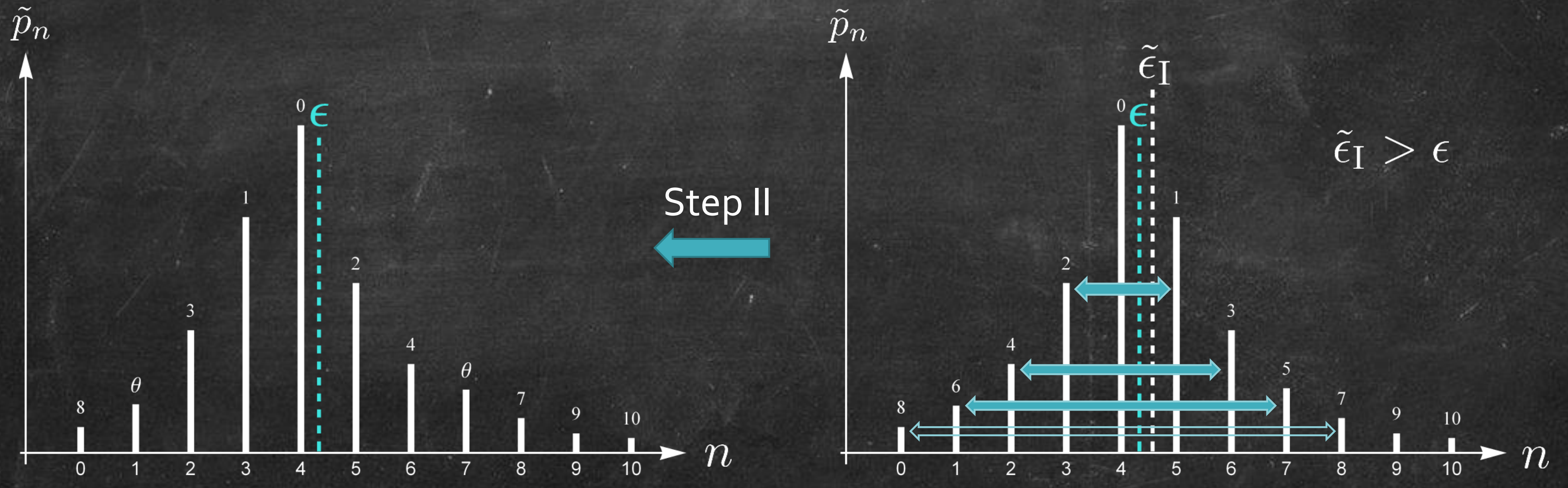
Minimal $\tilde{V}_\epsilon$ [mean square deviation from ϵ] but not the right average: $\tilde{\epsilon}_{\mathrm{I}} \neq \epsilon$ Step II

N. Friis and M. Huber, Quantum 2, 61 (2018) [arXiv:1708.00749]

Optimal Precision Charging

Optimal strategy has 2 Steps: Step II:

- Identify level pairs to adjust energy correctly
- Rotate between levels, starting with minimal $\frac{\Delta \tilde{V}_{\epsilon}}{|\Delta \tilde{\epsilon}|}$



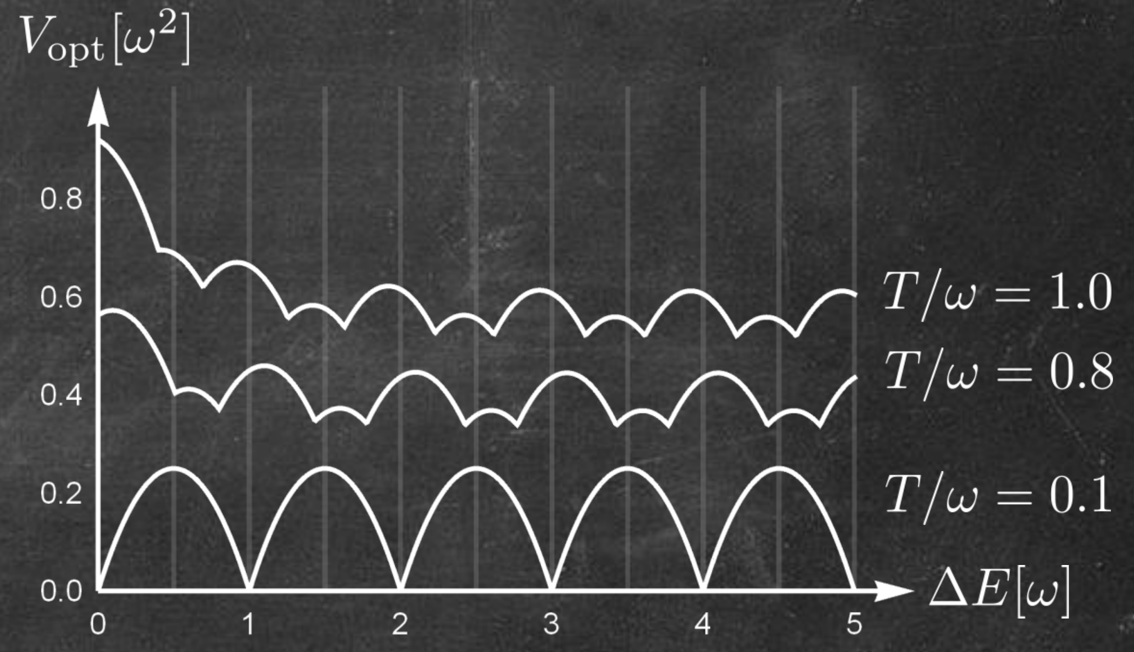
➡ Stop when reaching $\tilde{\epsilon}_{\mathrm{I}} = \epsilon$ ➡ $\tilde{V}_{\epsilon} = V(\rho) = V_{\mathrm{opt}}$

N. Friis and M. Huber, Quantum 2, 61 (2018) [arXiv:1708.00749]

Optimal Precision Charging

Single-mode batteries

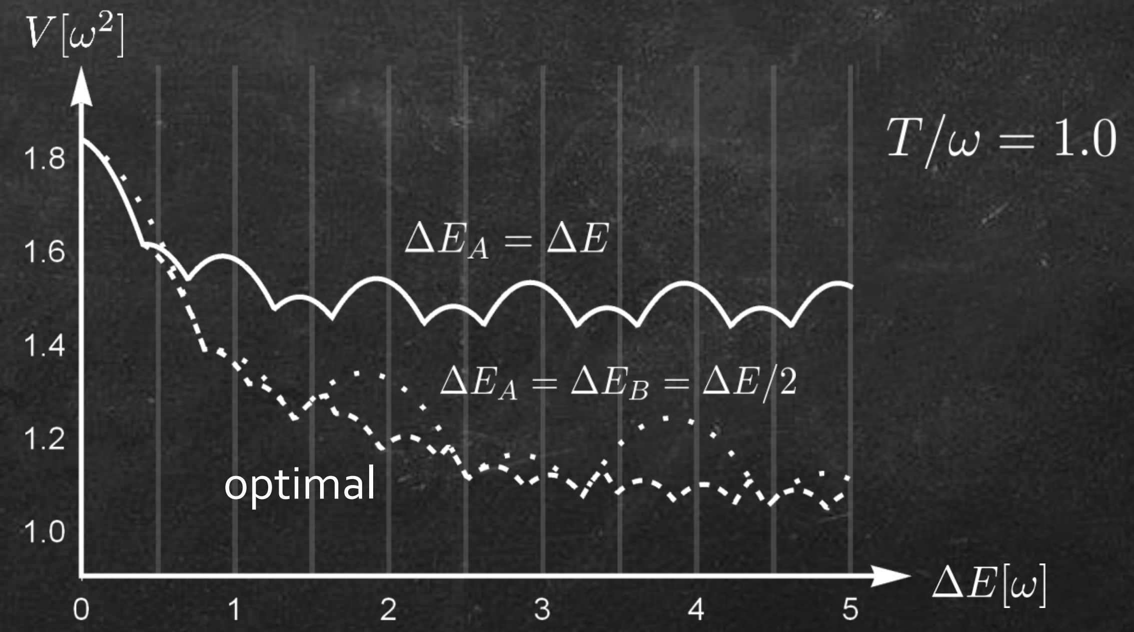
- For $T > 0$: variance may decrease
- For fixed T: $V_{\text{opt}}(\Delta E)$ bounded by constants



Multi-mode batteries

- Already local unitaries provide advantage
- Correlations can occur during step II

➡ Correlations can help but play no central role

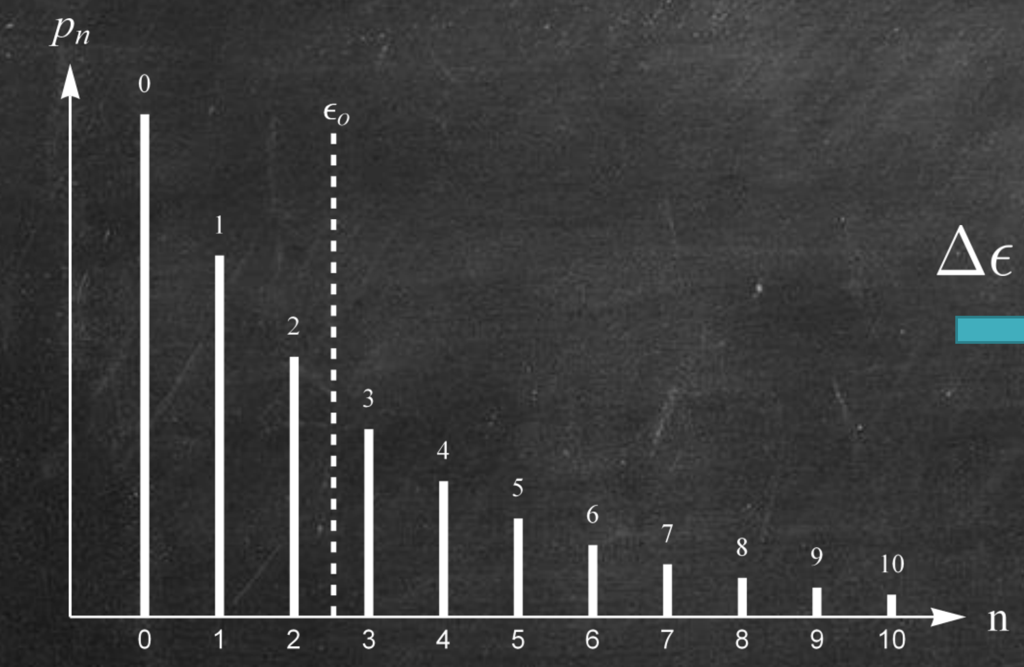


N. Friis and M. Huber, [Quantum 2, 61 (2018)] [arXiv:[1708.00749]]

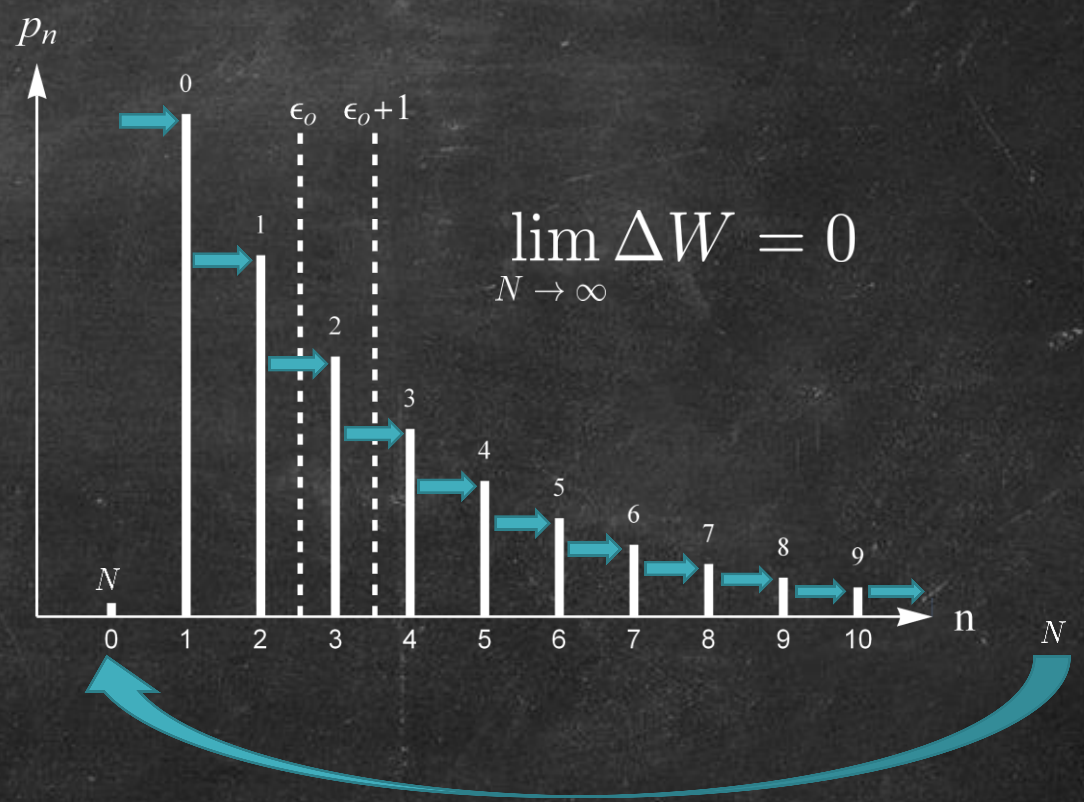
Minimal Fluctuations

For integer multiples of ω : $\Delta W = 0$

When $\Delta\epsilon = m \in \mathbb{N}$ ➡ Shift by m to the right



$\Delta\epsilon = 1$



$$\lim_{N\to\infty} \Delta W = 0$$

For non-integer $\Delta\epsilon$:

- Start shifting at $k = \lceil (\beta\omega)^{-1} \ln(1/\Delta\epsilon) \rceil > 0$
- Fine-tune: rotation between $k-1$ and k ➡

$$(\Delta W)^2 = (\Delta E - \lfloor \Delta E \rfloor)(\lceil \Delta E \rceil - \Delta E)$$
$$= V_{\mathrm{opt}}(T=0)$$

Gaussian Battery Charging

Limitation of Gaussian Unitaries?

Phase space description: Wigner representation $\rho \mapsto \mathcal{W}(x,p) = \frac{1}{(2\pi)^N} \int dy\, e^{-i\,p\,y} \langle x + \frac{y}{2} | \rho | x - \frac{y}{2} \rangle$

Observables: $\langle \hat{G} \rangle_\rho = \mathrm{Tr}(\hat{G}\rho) = \int dx\,dp\, \mathcal{W}(x,p)\, g(x,p)$ with $g(x,p) = \int dy\, e^{i\,p\,y} \langle x - \frac{y}{2} | \hat{G} | x + \frac{y}{2} \rangle$

Gaussian states $\mathcal{W}(\xi) = \frac{1}{\pi^N \sqrt{\det(\Gamma)}} \exp\left[-(\xi - \overline{\mathbb{X}})^T \Gamma^{-1} (\xi - \overline{\mathbb{X}})\right]$ $\overline{\mathbb{X}} = \langle \mathbb{X} \rangle_\rho, \quad \xi = (x_1, p_1, \ldots, x_N, p_N)^T$

Energy: $\frac{E(\rho)}{\omega} = \frac{1}{4}\left[\mathrm{Tr}(\Gamma) - 2\right] + \frac{1}{2}\|\overline{\mathbb{X}}\|^2$ Variance: $(\frac{\Delta \hat{H}}{\omega})^2 = \frac{1}{2}\overline{\mathbb{X}}^T \Gamma\, \overline{\mathbb{X}} + \frac{1}{8}\left[\mathrm{Tr}(\Gamma^2) - 2\right]$

Example: pure displacement $D(\alpha)$

$$\frac{\Delta E}{\omega} = \frac{1}{2}\|\overline{\mathbb{X}}\|^2 = \frac{1}{2}|\alpha|^2$$

$$(\frac{\Delta \hat{H}}{\omega})^2 = \frac{1}{2}\coth(\frac{\beta\omega}{2})\|\overline{\mathbb{X}}\|^2 + \frac{V(\tau)}{\omega^2}$$

as $\boxed{\Delta E \to \infty : \; V(\rho)/\Delta E \to \text{const.}}$

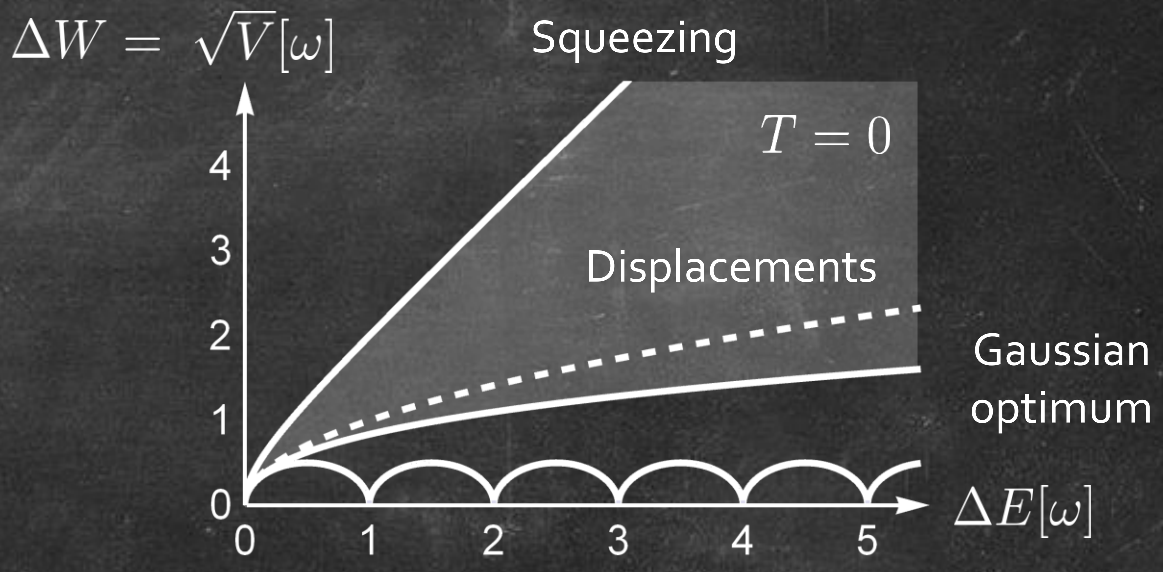
General Gaussian unitaries

- **Optimal**: combination of squeezing & displacement

  as $\boxed{\Delta E \to \infty : \; V(\rho)/\Delta E \to 0}$

- **Worst case**: pure single-mode squeezing

Optimal and Worst Single-Mode Gaussian strategies



Precision (variance):

- **Worst case**: pure single-mode squeezing
- **Optimal**: combination of squeezing & displacement
  as $\Delta E \to \infty$: $V(\rho)/\Delta E \to 0$
- **Pure displacement**: $V(\rho)/\Delta E \to \text{const.}$

Fluctuations:

- **Optimal**: combination of squeezing & displacement: as $\Delta E \to \infty$: $(\Delta W)^2/\Delta E \to 0$
- **Worst case**: in general also combination of squeezing & displacement

N. Friis and M. Huber, Quantum **2**, 61 (2018) [arXiv:1708.00749]

Measurement Cost of Quantum Measurements

Measurements can do work ➡ But what is the energy cost of performing the measurement?

Simple measurement model: $\rho_S \otimes \rho_P \mapsto U \rho_S \otimes \rho_P U^\dagger = \tilde{\rho}_{SP}$

(unknown) system state Pointer

Paper: Y. Guryanova, NF, M. Huber
[arXiv:1805.11899]

complete set of projectors: Π_i with $\Pi_i \Pi_j = \delta_{ij} \Pi_i$ for each $|i\rangle_S$

Ideal measurement is:

(i) *unbiased* $\mathrm{Tr}[\mathbb{I} \otimes \Pi_i \tilde{\rho}_{SP}] = \mathrm{Tr}[|i\rangle\langle i|_S \, \rho_S] = \rho_{ii} \quad \forall i$

(ii) *faithful* $C(\tilde{\rho}_{SP}) := \sum_i \mathrm{Tr}[|i\rangle\langle i| \otimes \Pi_i \, \tilde{\rho}_{SP}] = 1$

(ii) *non-invasive* $\mathrm{Tr}[|i\rangle\langle i|_S \, \tilde{\rho}_S] = \mathrm{Tr}[|i\rangle\langle i|_S \, \rho_S] = \rho_{ii} \quad \forall\, i.$

But: (ii) cannot be (exactly) satisfied if ρ_P has full rank (in particular, for finite-resource preparation)

➡ Non-ideal measurement satisyfing (i) possible: ➡ Energy cost for high values of $C(\tilde{\rho}_{SP})$

Summary and Remarks

- Work extraction using Gaussian operations ➡ Gaussian passivity
- Characterization of GP states using 1st & 2nd moments only
  ➡ provides protocol for Gaussian work extraction
- Non-Gaussian states: extractable work may be left (max. gap)
- Precision & Fluctuations for charging ➡ optimal general protocols
- Gaussian Operations ➡ non-optimal but good performance
- (Some) proofs rely on ∞-dim Hilbert space
- Finite energy cost of non-ideal quantum measurements

Papers: E. G. Brown, N. Friis, and M. Huber, New J. Phys. **18**, 113028 (2016) [arXiv:1608.04977].
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Y. Guryanova, N. Friis, and M. Huber [arXiv:1805.11899]

Thank you for your attention