## I@I

# The Thermodynamics of Creating Correlations 

Limitations and Optimal Protocols

## Group Retreat 2014

Maria Waldrast, September $28^{\text {th }}$ - October $1^{\text {st }}, 2014$

## Nicolai Friis

Institute for Quantum Optics and Quantum Information - Innsbruck work in collaboration with David E. Bruschi (Jerusalem), Martí Perarnau-Llobet, Karen V. Hovhannisyan, and Marcus Huber (Barcelona)

Optimal creation of total correlations
Optimal creation of entanglement

## Motivation - Quantum Thermodynamics

## Some approaches to thermodynamics in the quantum domain

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- Landauer's principle: minimum energy for bit erasure (see, e.g., Reeb \& Wolf, arXiv:1306.4352; Esposito \& Van den Broeck, Europhys. Lett. 95, 40004 (2011) [arXiv:1104.5165])


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## Connection between resource theories?

thermodynamics $\longleftrightarrow$ quantum information


## Outline

## We ask ${ }^{1}$

How strongly can one correlate a system at initial temperature $T$ if the work $W$ is supplied?

- Cost of correlating isolated systems of qubits (Huber et al., arXiv:1404.2169)
- Bound for overall correlations measured by the mutual information
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- Optimal entanglement generation between two bosonic modes
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## Optimal creation of total correlations

## Bound on optimal correlations

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For arbitrarily large bath $\Delta F_{B}=0, \mathcal{I}_{S B}=0$ achievable (see Åberg, Nat. Commun. 4, 1925 (2013) [arXiv:1110.6121])

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Overall work cost:
$W=W_{\mathrm{I}}+W_{\mathrm{II}}$

$$
\begin{aligned}
= & E\left(\tau_{S}\left(\beta_{\mathrm{I}}\right)\right)-E\left(\tau_{S}(\beta)\right)-T\left[S\left(\tau_{S}\left(\beta_{\mathrm{I}}\right)\right)-S\left(\tau_{S}(\beta)\right)\right] \\
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## Optimal protocol for the generation of mutual information

Without loss of generality we may split the protocol into two steps


Overall work cost:

$$
\begin{aligned}
W & =W_{\mathrm{I}}+W_{\mathrm{II}} \\
& =E\left(\tau_{S}\left(\beta_{\mathrm{I}}\right)\right)-E\left(\tau_{S}(\beta)\right)-T\left[S\left(\tau_{S}\left(\beta_{\mathrm{I}}\right)\right)-S\left(\tau_{S}(\beta)\right)\right] \\
& +E\left(\tau_{S}\left(\beta_{\mathrm{II}}\right)\right)-E\left(\tau_{S}\left(\beta_{\mathrm{I}}\right)\right)
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$$

On the other hand

$$
\mathcal{I}_{S_{1} S_{2}}=S\left(\tau_{S_{1}}\left(\beta_{\mathrm{II}}\right)\right)+S\left(\tau_{S_{2}}\left(\beta_{\mathrm{II}}\right)\right)-S\left(\rho_{S}^{\mathrm{II}}\right)
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S\left(\tau_{S}\left(\beta_{\mathrm{I}}\right)\right)=S\left(\tau_{S}\left(\beta_{\mathrm{II}}\right)\right)-\mathcal{I}_{S_{1} S_{2}}
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\end{aligned}
$$

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S\left(\tau_{S}\left(\beta_{\mathrm{I}}\right)\right)=S\left(\tau_{S}\left(\beta_{\mathrm{II}}\right)\right)-\mathcal{I}_{S_{1} S_{2}}
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& =\Delta F_{S}\left(\beta \rightarrow \beta_{\mathrm{II}}\right)+T \mathcal{I}_{S_{1} S_{2}}
\end{aligned}
$$

## Optimal protocol for the generation of mutual information

Without loss of generality we may split the protocol into two steps


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& =T S\left(\tau_{S}\left(\beta_{\mathrm{II}}\right) \| \tau_{S}(\beta)\right)+T \mathcal{I}_{S_{1} S_{2}}
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\end{aligned}
$$

Optimal conversion of energy into correlations when

$$
\beta_{\mathrm{II}}=\beta, \quad T_{\mathrm{II}}=T \quad \Rightarrow \quad S\left(\tau_{S}\left(\beta_{\mathrm{II}}\right) \| \tau_{S}(\beta)\right)=0
$$

## Optimal protocol for the generation of mutual information

Without loss of generality we may split the protocol into two steps


Optimal protocol: distinguish two regimes

Low energy
High energy: $T_{\Pi}>T, \quad T_{I}=0$, ground state reached in step I

## Optimal protocol for the generation of mutual information

Without loss of generality we may split the protocol into two steps


Optimal protocol: distinguish two regimes
Low energy: $\quad T_{\text {II }}=T, \quad T_{\mathrm{I}}>0$,

[^6]
## Optimal protocol for the generation of mutual information

Without loss of generality we may split the protocol into two steps


Optimal protocol: distinguish two regimes
Low energy: $\quad T_{\text {II }}=T, \quad T_{\mathrm{I}}>0$,
High energy: $\quad T_{\text {II }}>T, \quad T_{\mathrm{I}}=0$, ground state reached in step I

Threshold energy:
free energy difference to ground state

## Optimal protocol for the generation of mutual information

Without loss of generality we may split the protocol into two steps


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Low energy: $\quad T_{\text {II }}=T, \quad T_{\mathrm{I}}>0$,
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Threshold energy: $W=T S\left(\tau_{S}(\beta)\right)=-F\left(\tau_{S}(\beta)\right)+E\left(\tau_{S}(\beta)\right)$

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W_{\mathrm{I}}=-F\left(\tau_{S}(\beta)\right) \geq 0 \quad \text { free energy difference to ground state }
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$$
\begin{gathered}
W_{\mathrm{I}}=-F\left(\tau_{S}(\beta)\right) \geq 0 \quad \text { free energy difference to ground state } \\
W_{\mathrm{II}}=E\left(\tau_{S}(\beta)\right) \quad \text { supplies energy for } \beta_{\mathrm{II}}=\beta
\end{gathered}
$$

## Optimal protocol for the generation of mutual information

Without loss of generality we may split the protocol into two steps


Optimal protocol: distinguish two regimes

High energy:
$\beta_{\Pi}$ implicitly given by $E\left(\tau_{S}\left(\beta_{\Pi}\right)\right)=W+F\left(\tau_{S}(\beta)\right)$

## Optimal protocol for the generation of mutual information

Without loss of generality we may split the protocol into two steps


Optimal protocol: distinguish two regimes

$$
\text { Low energy: } \quad \mathcal{I}_{S_{1} S_{2}}=\beta W \text { if } W \leq \operatorname{TS}\left(\tau_{S}(\beta)\right) \text {, }
$$

## Optimal protocol for the generation of mutual information

Without loss of generality we may split the protocol into two steps


Optimal protocol: distinguish two regimes

Low energy: $\quad \mathcal{I}_{S_{1} S_{2}}=\beta W$ if $W \leq \operatorname{TS}\left(\tau_{S}(\beta)\right)$,
High energy: $\mathcal{I}_{S_{1} S_{2}}=S\left(\tau_{S}\left(\beta_{\text {II }}\right)\right)$ if $\quad W>\operatorname{TS} S\left(\tau_{S}(\beta)\right)$,

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## High energy regime: $W>T S\left(\tau_{s}(\beta)\right)$

## 2 fermionic modes at frequency $\omega$

Fermi-Dirac statistics: partition function $\mathcal{Z}_{\mathrm{FD}}=1+e^{-\beta}$
Thermal state: average particle number
Pauli exclusion principle: $0 \leq N_{S_{i}} \leq 1$ for any state

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}, T=$

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$\Rightarrow$ maximally useful energy for protocol: $W_{\max }=2 T \ln \left(e^{\beta}+1\right)-\omega$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}, T=0,0.1, \ldots, 0.9,1$

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## 2 bosonic modes at frequency $\omega$

Bose-Einstein statistics: partition function $\mathcal{Z}_{\mathrm{BE}}=\left(1-e^{-\beta}\right)^{-1}$
Thermal state: average particle number

Energy of thermal state: $E(\tau$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## High energy regime: $\quad W>T S\left(\tau_{S}(\beta)\right)$

## 2 bosonic modes at frequency $\omega$

Bose-Einstein statistics: partition function $\mathcal{Z}_{\mathrm{BE}}=\left(1-e^{-\beta}\right)^{-1}$
Thermal state: average particle number $N_{S_{1}}=N_{S_{2}}=\left(e^{\beta}-1\right)^{-1}$

```
Energy of thermal state: E (\tau
```

Entropy of thermal state:

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

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Energy of thermal state: $E\left(\tau_{S}(\beta)\right)=\omega[\operatorname{coth}(\beta / 2)-1]$
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$$
\text { where } f(x)=\frac{x+1}{2} \ln \left(\frac{x+1}{2}\right)-\frac{x-1}{2} \ln \left(\frac{x-1}{2}\right)
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$$

For $W \gg T S\left(\tau_{S}(\beta)\right)$ : all energy of step II for correlations

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

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$$

For $W \gg T S\left(\tau_{S}(\beta)\right)$ : all energy of step II for correlations

$$
W_{\mathrm{II}}=E\left(\tau_{S}\left(\beta_{\mathrm{II}}\right)\right) \text { and } \mathcal{I}_{S_{1} S_{2}}=S\left(\tau_{S}\left(\beta_{\mathrm{II}}\right)\right)
$$

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$$

For $W \gg T S\left(\tau_{S}(\beta)\right)$ : all energy of step II for correlations

$$
\begin{gathered}
W_{\text {II }}=E\left(\tau_{S}\left(\beta_{\text {II }}\right)\right) \text { and } \mathcal{I}_{S_{1} S_{2}}=S\left(\tau_{S}\left(\beta_{\text {II }}\right)\right) \\
\mathcal{I}_{S_{1} S_{2}}=2+2 \ln \left(\frac{1}{2} \frac{W_{\text {II }}}{\omega}\right)+\mathcal{O}\left(\frac{\omega}{W_{\text {II }}}\right)
\end{gathered}
$$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## Optimal creation of entanglement

## Entanglement of Formation for two fermionic modes

## Superselected entanglement of formation

$$
E_{o F}=\inf _{\mathcal{D}\left(\rho_{S}\right)} \sum_{i} p_{i} \mathcal{E}\left(\left|\psi_{i}\right\rangle\right)
$$

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\text { and } \left.\mathcal{D}\left(\rho_{S}\right)=\left\{p_{i},\left|\psi_{i}\right\rangle\left|\sum_{i} p_{i}\right| \psi_{i}\right\rangle\left\langle\psi_{i}\right|=\rho_{S}\right\}
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\end{gathered}
$$

superselection rules: take infimum only over allowed states $\left|\psi_{i}\right\rangle$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## Entanglement of Formation for two fermionic modes

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E_{o F}=\inf _{\mathcal{D}\left(\rho_{S}\right)} \sum_{i} p_{i} \mathcal{E}\left(\left|\psi_{i}\right\rangle\right) \\
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\text { and } \left.\mathcal{D}\left(\rho_{S}\right)=\left\{p_{i},\left|\psi_{i}\right\rangle\left|\sum_{i} p_{i}\right| \psi_{i}\right\rangle\left\langle\psi_{i}\right|=\rho_{S}\right\}
\end{gathered}
$$

superselection rules: take infimum only over allowed states $\left|\psi_{i}\right\rangle$
Here: no superpositions of even and odd fermions numbers

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## Entanglement of Formation for two fermionic modes

## Superselected entanglement of formation

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\begin{gathered}
E_{o F}=\inf _{\mathcal{D}\left(\rho_{S}\right)} \sum_{i} p_{i} \mathcal{E}\left(\left|\psi_{i}\right\rangle\right) \\
\text { where } \mathcal{E}\left(\left|\psi_{i}\right\rangle\right)=S\left(\operatorname{Tr}_{S_{2}}\left(\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|\right)\right) \\
\text { and } \left.\mathcal{D}\left(\rho_{S}\right)=\left\{p_{i},\left|\psi_{i}\right\rangle\left|\sum_{i} p_{i}\right| \psi_{i}\right\rangle\left\langle\psi_{i}\right|=\rho_{S}\right\}
\end{gathered}
$$

superselection rules: take infimum only over allowed states $\left|\psi_{i}\right\rangle$
Here: no superpositions of even and odd fermions numbers
With this: even and odd subspace decouple: $\theta_{\text {even }}$ and $\theta_{\text {odd }}$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## Entanglement of Formation for two fermionic modes

## Superselected entanglement of formation

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$$
E_{o F}=\ln (2)\left[\left|1-N_{S_{1}}^{\mathrm{I}}-N_{S_{2}}^{\mathrm{I}}\right| \sin \left(2 \theta_{\text {even }}\right)+\left|N_{S_{1}}^{\mathrm{I}}-N_{S_{2}}^{\mathrm{I}}\right| \sin \left(2 \theta_{\text {odd }}\right)\right]
$$

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\text { energy cost: } \frac{W_{\mathrm{II}}}{\omega}=2\left(1-N_{S_{1}}^{\mathrm{I}}-N_{S_{2}}^{\mathrm{I}}\right) \sin ^{2}\left(\theta_{\text {even }}\right)
\end{gathered}
$$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## Entanglement of Formation for two fermionic modes

## Superselected entanglement of formation

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\end{gathered}
$$

$$
\text { numerical optimization over } \theta_{\text {even }}, N_{S_{1}}^{\mathrm{I}}, N_{S_{2}}^{\mathrm{I}}
$$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## Entanglement of Formation for two fermionic modes

## Previous Protocol




## Optimal Protocol

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}, T=0,0.1, \ldots, 0.9,1$

## Entanglement of Formation for two fermionic modes

## Previous Protocol




## Optimal Protocol




Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}, T=0,0.1, \ldots, 0.9,1$ and $\infty$

## State description for two bosonic modes

two harmonic oscillators: Hilbert space infinite-dimensional $\Rightarrow$ restrict problem

# Fully described by $1^{\text {st }}$ and $2^{\text {nd }}$ moments: expectations values of quadratures $\mathbb{X}_{n}$ 



Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## State description for two bosonic modes

two harmonic oscillators: Hilbert space infinite-dimensional $\Rightarrow$ restrict problem

## Two-mode Gaussian states

Fully described by $1^{\text {st }}$ and $2^{\text {nd }}$ moments: expectations values of quadratures

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## State description for two bosonic modes

two harmonic oscillators: Hilbert space infinite-dimensional $\Rightarrow$ restrict problem

## Two-mode Gaussian states

Fully described by $1^{\text {st }}$ and $2^{n d}$ moments: expectations values of quadratures $\mathbb{X}_{n}$

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## State description for two bosonic modes

two harmonic oscillators: Hilbert space infinite-dimensional $\Rightarrow$ restrict problem

## Two-mode Gaussian states

Fully described by $1^{\text {st }}$ and $2^{\text {nd }}$ moments: expectations values of quadratures $\mathbb{X}_{n}$

$$
\mathbb{X}_{(2 n-1)}=\frac{1}{\sqrt{2}}\left(a_{n}+a_{n}^{\dagger}\right) \text { and } \mathbb{X}_{(2 n)}=\frac{-i}{\sqrt{2}}\left(a_{n}-a_{n}^{\dagger}\right)
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First moments: $\operatorname{Tr}\left(\mathbb{X}_{n} \rho_{S}\right)=$ irrelevant for entanglement

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two harmonic oscillators: Hilbert space infinite-dimensional $\Rightarrow$ restrict problem

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X_{(2 n-1)}=\frac{1}{\sqrt{2}}\left(a_{n}+a_{n}^{\dagger}\right) \quad \text { and } X_{(2 n)}=\frac{-i}{\sqrt{2}}\left(a_{n}-a_{n}^{\dagger}\right)
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First moments: $\operatorname{Tr}\left(\mathbb{X}_{n} \rho_{S}\right)=0$ irrelevant for entanglement Second moments: $\left(\sigma_{S}\right)_{m n}=\operatorname{Tr}\left(\left\{\mathbb{X}_{m}, \mathbb{X}_{n}\right\} \rho_{S}\right)$ covariance matrix

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## State description for two bosonic modes

two harmonic oscillators: Hilbert space infinite-dimensional $\Rightarrow$ restrict problem

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Energy: $E\left(\sigma_{S}\right)=\omega(\nu(T)-1), \quad$ Entropy: $S\left(\sigma_{S}\right)=2 f(\nu(T))$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

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Energy: $E\left(\sigma_{S}\right)=\omega(\nu(T)-1)$, Entropy: $S\left(\sigma_{S}\right)=2 f(\nu(T))$

$$
\text { where } f(x)=\frac{x+1}{2} \ln \left(\frac{x+1}{2}\right)-\frac{x-1}{2} \ln \left(\frac{x-1}{2}\right)
$$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## State description for two bosonic modes

two harmonic oscillators: Hilbert space infinite-dimensional $\Rightarrow$ restrict problem

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$$

Energy for cooling: $\frac{W_{\mathrm{I}}}{\omega}=\nu\left(T_{\mathrm{I}}\right)-\nu(T)-2 \beta^{-1}\left[f\left(\nu\left(T_{\mathrm{I}}\right)\right)-f(\nu(T))\right]$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## Entanglement of Formation for two bosonic modes

## Entanglement for two-mode Gaussian states

Necessary and sufficient condition: negative partial transpose (NPT)

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## Entanglement of Formation for two bosonic modes

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Necessary and sufficient condition: negative partial transpose (NPT)
Translates to: entanglement if and only if $0 \leq \tilde{\nu}_{-}<1$,

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$\qquad$

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$\qquad$

## Entanglement of Formation for two bosonic modes

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Optimal Gaussian entangling operation: state remains symmetric: $E_{o F}$ computable

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E_{o F}= \begin{cases}\mathfrak{h}\left(\tilde{\nu}_{-}\right), & \text {if } 0 \leq \tilde{\nu}_{-}<1 \\ 0, & \text { if } \tilde{\nu}_{-} \geq 1\end{cases}
$$

with $\mathfrak{h}(x)=h_{+}(x) \ln \left(h_{+}(x)\right)-h_{-}(x) \ln \left(h_{-}(x)\right)$, and $h_{ \pm}(x)=\frac{(x \pm 1)^{2}}{4 x}$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

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\frac{W_{\mathrm{II}}}{\omega}=\frac{\nu^{2}\left(T_{\mathrm{I}}\right)}{2 \tilde{\nu}_{-}}\left[\frac{\tilde{\nu}_{-}}{\nu\left(T_{\mathrm{I}}\right)}-1\right]^{2}
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Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## Entanglement of Formation for two bosonic modes

## Entanglement for two-mode Gaussian states

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\end{gathered}
$$

$$
\frac{W_{\mathrm{II}}}{\omega}=\frac{\nu^{2}\left(T_{\mathrm{I}}\right)}{2 \tilde{\nu}_{-}}\left[\frac{\tilde{\nu}_{-}}{\nu\left(T_{\mathrm{I}}\right)}-1\right]^{2} \Rightarrow \text { numerically optimize over } T_{\mathrm{I}}
$$

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}$

## Entanglement of Formation for two bosonic modes

## Gaussian Protocol




## Optimal Protocols

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}, T=0,0.1, \ldots, 0.9,1$

## Entanglement of Formation for two bosonic modes

## Gaussian Protocol



## Optimal Protocols



Outperformed by non-Gaussian
protocol:

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}, T=0,0.1, \ldots, 0.9,1$

## Entanglement of Formation for two bosonic modes

## Gaussian Protocol



## Optimal Protocols



Gaussian protocol optimal

Outperformed by non-Gaussian
protocol
Rotation to "Bell state" in 2-dim
subspace

Notation: $\beta=\omega / T$, temperatures specified in units of $\hbar \omega / k_{\mathrm{B}}, T=0,0.1, \ldots, 0.9,1$

## Entanglement of Formation for two bosonic modes

## Gaussian Protocol



## Optimal Protocols



Gaussian protocol optimal
Low energies:
Outperformed by non-Gaussian protocol:

Rotation to "Bell state" in 2-dim

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## Entanglement of Formation for two bosonic modes

## Gaussian Protocol



## Optimal Protocols



Gaussian protocol optimal
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## Thank you for your attention.

D. E. Bruschi, M. Perarnau-Llobet, N. Friis, K. V. Hovhannisyan, and M. Huber, arXiv:1409.4647 [quant-ph].


[^0]:    Outine of our contribution
    Generating total correlations:

    - Bound for overall correlations measured by the mutual information
    - Optimal protocol saturating this bound

    Generating entanglement - bottom-up approach:

    - Ontimal entanglement generation between two bosonic modes
    - Optimal entanglement generation between two fermionic modes
    ${ }^{1}$ D. E. Bruschi, M. Perarnau-Llobet, N. Friis, K. V. Hovhannisyan, and M. Huber, arXiv:1409.4647 [quant-ph].

[^1]:    ${ }^{1}$ D. E. Bruschi, M. Perarnau-Llobet, N. Friis, K. V. Hovhannisyan, and M. Huber, arXiv:1409.4647 [quant-ph].

[^2]:    ${ }^{1}$ D. E. Bruschi, M. Perarnau-Llobet, N. Friis, K. V. Hovhannisyan, and M. Huber, arXiv:1409.4647 [quant-ph].

[^3]:    ${ }^{1}$ D. E. Bruschi, M. Perarnau-Llobet, N. Friis, K. V. Hovhannisyan, and M. Huber, arXiv:1409.4647 [quant-ph].

[^4]:    ${ }^{1}$ D. E. Bruschi, M. Perarnau-Llobet, N. Friis, K. V. Hovhannisyan, and M. Huber, arXiv:1409.4647 [quant-ph].

[^5]:    For arbitrarily arge bath $\Delta F_{R}=0, I_{S R}=0$ achievable

[^6]:    High energy: $T_{\Pi}>T, \quad I$
    ground state reached in step I

