Pure States and Projective Measurements

Convenient fictions in a thermodynamic world of incomplete knowledge

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Pure States and Projective Measurements

are central features of Quantum Mechanics (and hence of Quantum Information)

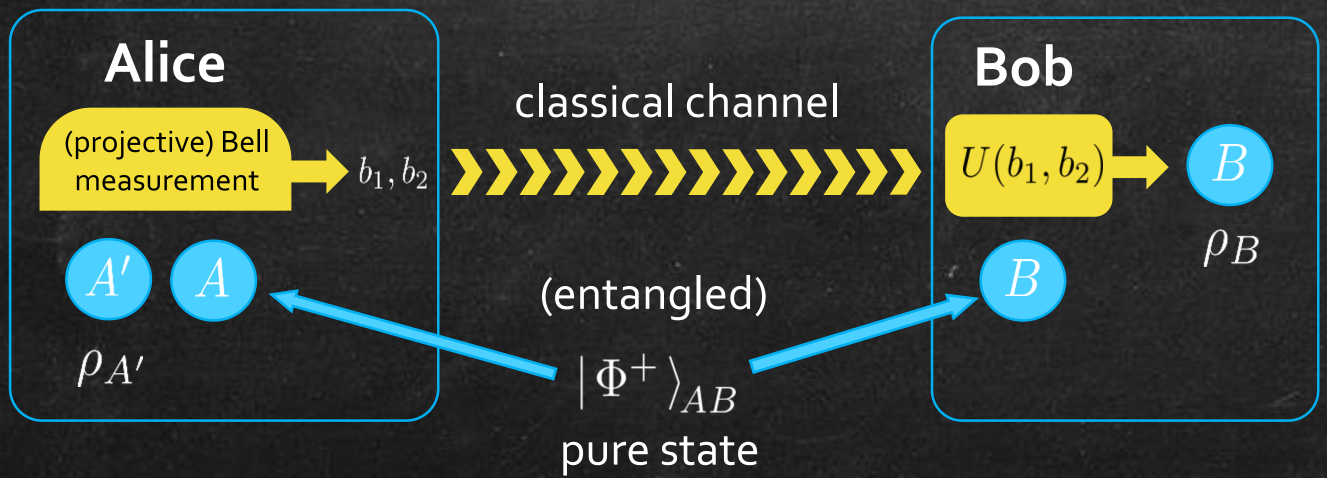
Pure states $|\Psi\rangle$ complete information about the system

Projective measurements observable $O = \sum_n \lambda_n |n\rangle\langle n|$

outcome "n" → system left in state $|n\rangle$ (projection postulate)

probability for outcome "n": $p_n = |\langle n|\Psi\rangle|^2$ (Born rule)

For instance, consider quantum teleportation



State ρ can be teleported perfectly from Alice to Bob, if:

- $|\Phi^+\rangle_{AB}$ is pure (max. entangled) state
- Bell measurement is (ideal) projective measurement
- $U(b_1, b_2)$ is unitary

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What if we do not have complete information?

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Mixed states ρ density operator (normalized $\mathrm{Tr}(\rho) = 1$, positive semidefinite $\rho \geq 0$)

Positive operator-valued measure (POVM) $\{M_n\}_n$ $M_n \geq 0$ $\sum_n M_n = \mathbb{1}$

Mixed states/POVMs can be viewed as pure states/projective measurements on a larger Hilbert space via Stinespring/Naimark dilation.

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How can we obtain complete information?

How can we prepare pure quantum states?

A Thermodynamic Perspective

"If physical theories were people, thermodynamics would be the village witch. Over the course of three centuries, she smiled quietly as other theories rose and withered, surviving major revolutions in physics, like the advent of general relativity and quantum mechanics. The other theories find her somewhat odd, somehow different in nature from the rest, yet everyone comes to her for advice, and no-one dares to contradict her." From page 1, Goold, Huber, Riera, del Rio, Skrzypczyk, J. Phys. A: Math. Theor. **49**, 143001 (2016)

Historically: Thermodynamics concerned with system with many degrees of freedom

Access only to (few) macroscopically observable parameters (energy, temperature, etc.)

Here: Thermodynamics as a theory based on incomplete information.

Starting point: systems in thermal equilibrium $\tau(\beta) = \frac{e^{-\beta H}}{\mathcal{Z}}$

Hamiltonian $H = \sum_n E_n |n\rangle\langle n|$ partition function $\mathcal{Z} = \mathrm{Tr}(e^{-\beta H})$ Inverse temperature $\beta = 1/(k_{\mathrm{B}}T)$

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Jayne's principle: maximal (von Neumann) entropy at fixed average energy

Converseley: minimal (average) energy at fixed entropy

From Thermodynamics to Quantum Information

Some of the earliest contributions to quantum information, e.g.,

R. P. Poplavskii, *Thermodynamic models of information processes* (translated title), Sov. Phys. Usp. **18**, 222 (1975)

Charles H. Bennett, *The thermodynamics of computation – a review*, Int. J. Theor. Phys. **21**, 905 (1982)

...reference thermodynamics.

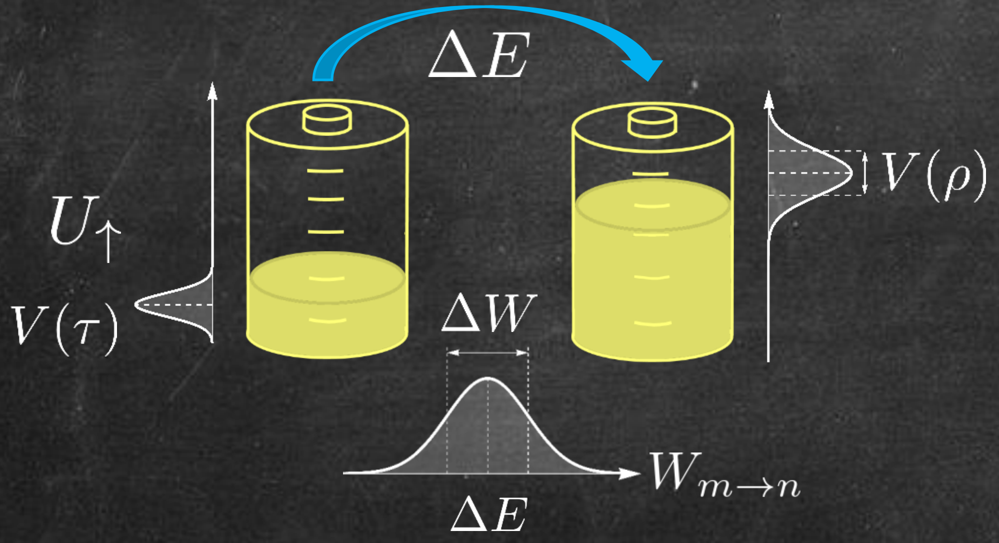
And even some work in classical computer science is highly influential to this day in (quantum) thermodynamics

Rolf Landauer, *Irreversibility and Heat Generation in the Computing Process*, IBM J. Res. Dev. **5**, 183 (1961).

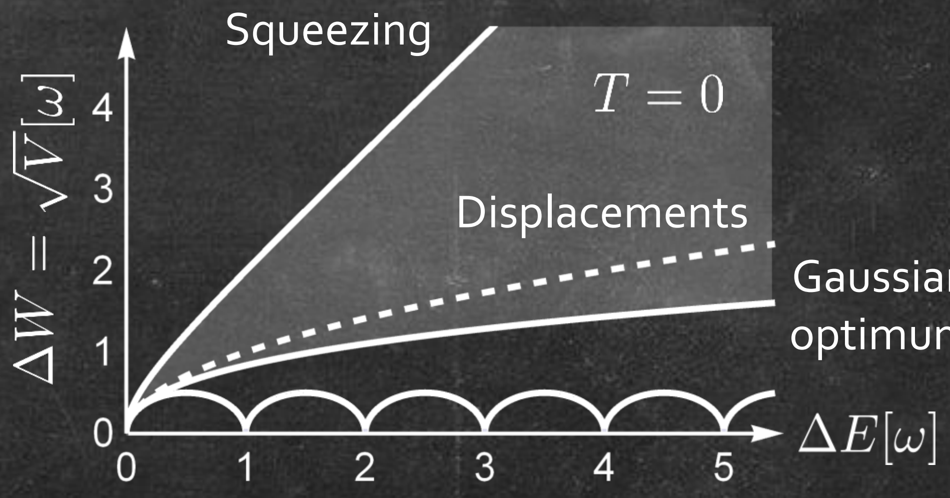
From Thermodynamics to Quantum Information

For us, thermal states $\tau(\beta) = \frac{e^{-\beta H}}{\mathcal{Z}}$ as starting point for various tasks:

Charging quantum batteries



Role of "complexity" of operations?



NF and Marcus Huber, *Precision and Work Fluctuations in Gaussian Battery Charging*, Quantum **2**, 61 (2018)

„Inverted problem": Work extraction via Gaussian operations

Eric Brown, NF & Marcus Huber, *Passivity and practical work extraction using Gaussian operations*, New J. Phys. **18**, 113028 (2016)

From Thermodynamics to Quantum Information

Optimal conversion of work into correlations

Uncorrelated initial state (thermal) $\rho_{AB}^{\mathrm{in}} = \tau_A(\beta) \otimes \tau_B(\beta)$

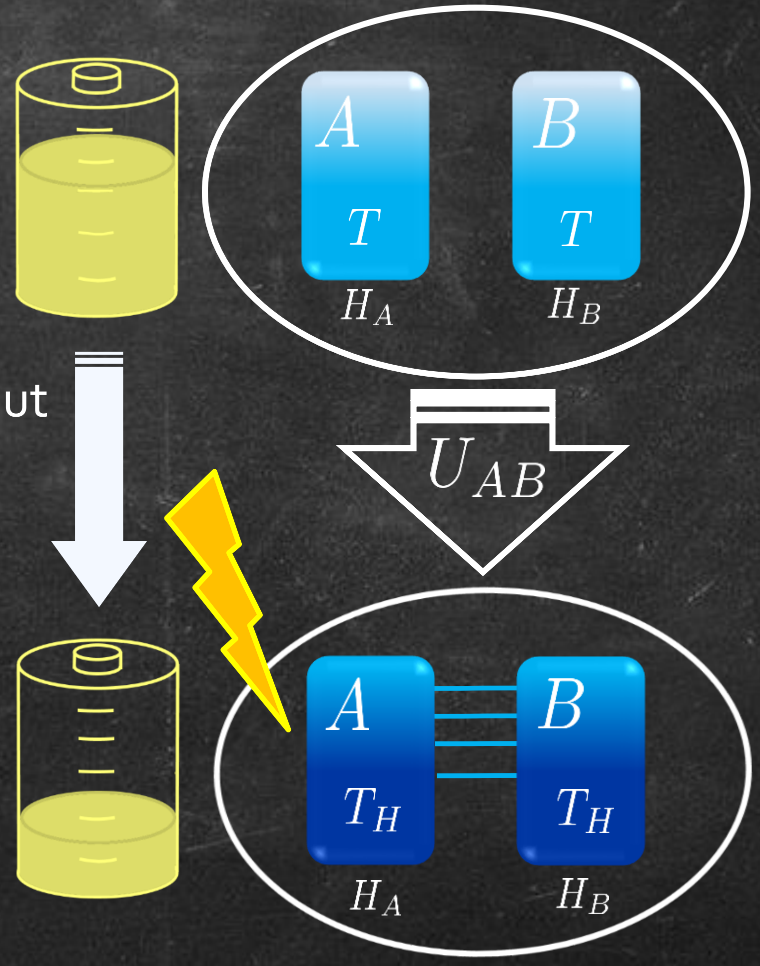
Cooling and (global unitary) correlating via U_{AB}

Correlations (mutual information)

$$\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

Energy input

ΔE



For symmetric systems $H_A = H_B$: $\boxed{\Delta\mathcal{I}(\rho_{AB}) \leq \beta \Delta E}$

"=" ⟷ $\exists$ symmetrically thermalizing unitaries (?)

F. Bakhshinezhad, F. Clivaz, G. Vitagliano, P. Erker, A. Rezakhani, M. Huber & NF, J. Phys. A: Math. Theor. **52**, 465303 (2019).

So how can we prepare a pure state?

Starting from a thermal state, we can… …"cool" the system

But what precisely does this mean? Lowering the temperature

Making the state "more pure"

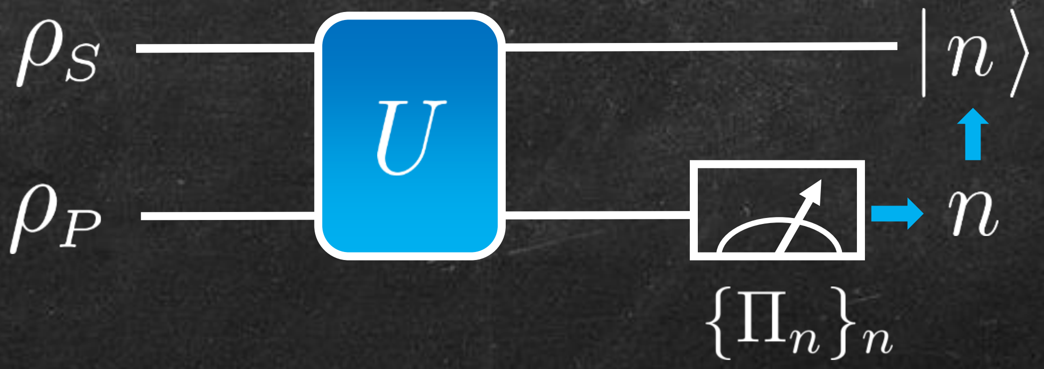
Gaining information about the (state of the) system

(In some sense) most information gained by an ideal projective measurement

Interaction between system and pointer

$$\rho_S \otimes \rho_P \longrightarrow \tilde{\rho}_{SP}$$

Followed by pointer read-out



Ideal Projective Measurements

Three important properties

Faithful $C(\tilde{\rho}_{SP}) := \sum_n \mathrm{Tr}\big[\,|n\rangle\langle n| \otimes \Pi_n\, \tilde{\rho}_{SP}\big] \,=\, 1$

"Correlation" between pointer outcome and post-measurement system state

Unbiased $\mathrm{Tr}\big[\mathbb{1} \otimes \Pi_n\, \tilde{\rho}_{SP}\big] = \mathrm{Tr}\big[\,|n\rangle\langle n|_S\, \rho_S\big] \,=\, \rho_{nn} \quad \forall n \;\&\; \forall \rho_S$

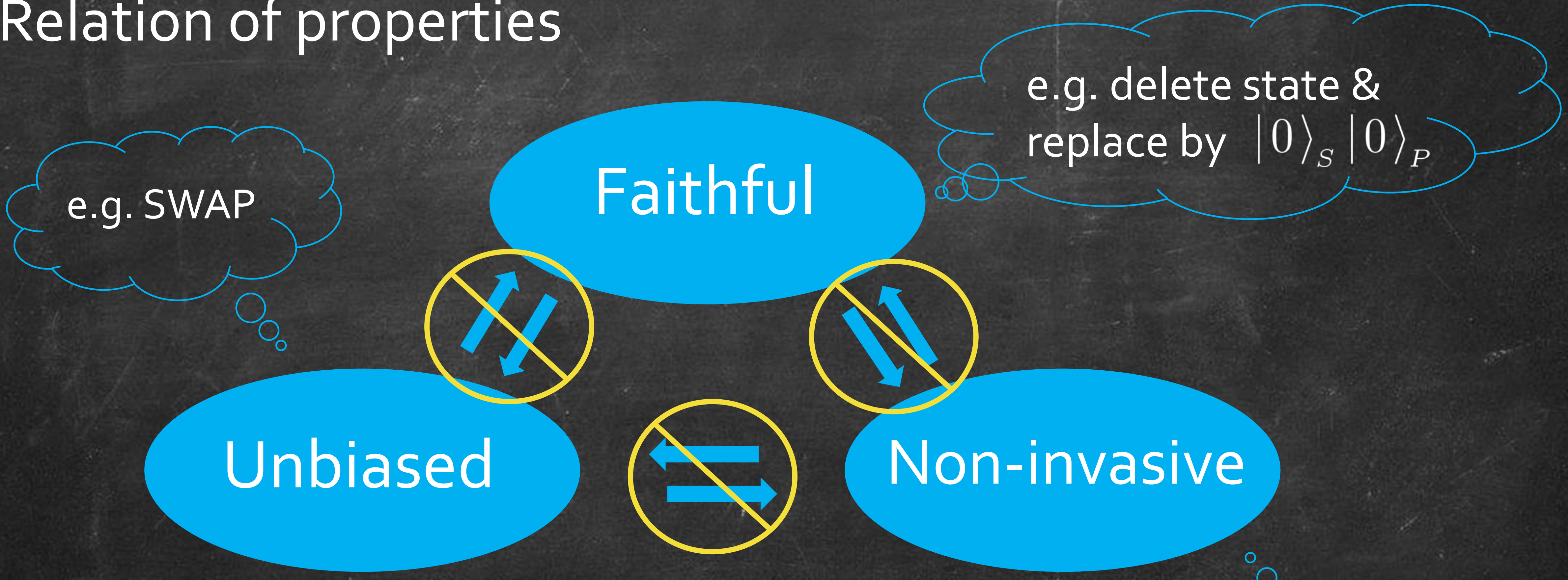
Pointer shows outcome statistics of pre-measurement system state

Non-invasive $\mathrm{Tr}\big[\,|n\rangle\langle n|_S\, \tilde{\rho}_{SP}\big] = \mathrm{Tr}\big[\,|n\rangle\langle n|_S\, \rho_S\big] \quad \forall n \;\&\; \forall \rho_S$

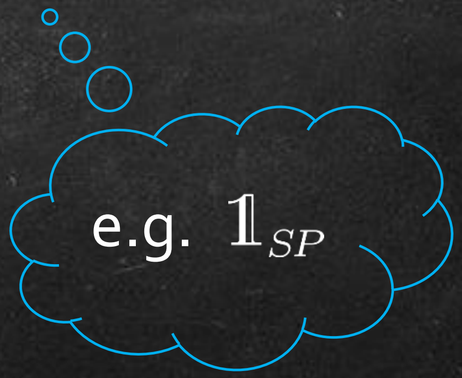
Minimal disturbance of pre-measurement system state

Yelena Guryanova, NF, and Marcus Huber, *Ideal projective measurements have infinite resource costs*, Quantum 4, 222 (2020)

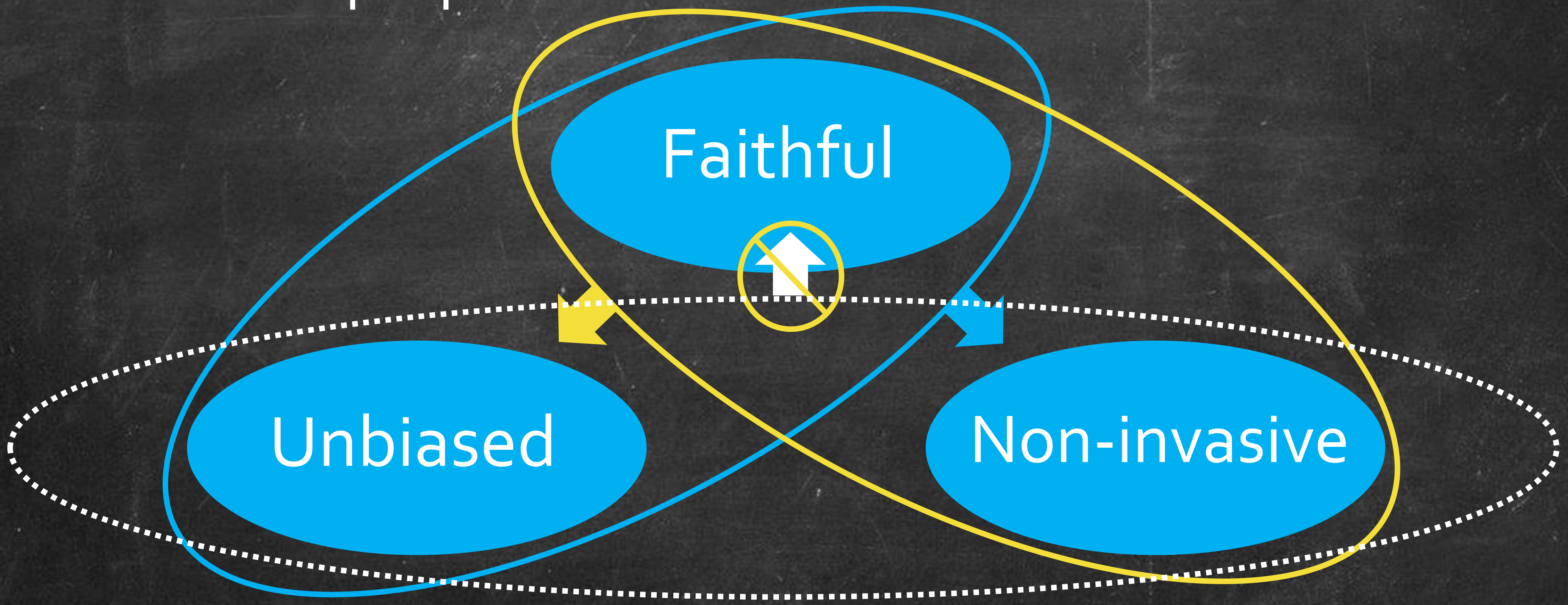
Relation of properties



None of the properties alone imply any of the other properties



Relation of properties



Faithfulness + second property ⟹ third property

Faithfulness (perfect correlation) requires non-full rank pointer state

Resource-theoretic view on quantum thermodynamics

Pointer initially thermal: $\rho_P = \tau_P(\beta)$

Energy/work required to move out ouf equilibrium

In particular: pointer has full rank

To achieve perfect correlations: cool to ground state $\beta \rightarrow \infty$

3rd Law (Nernst): ∞ energy or ∞ time or ∞ complexity*

* Operations with infinite interaction range, i.e., acting non-trivially on an infinite-dimensional subspace (we will come back to this)

Non-ideal measurements

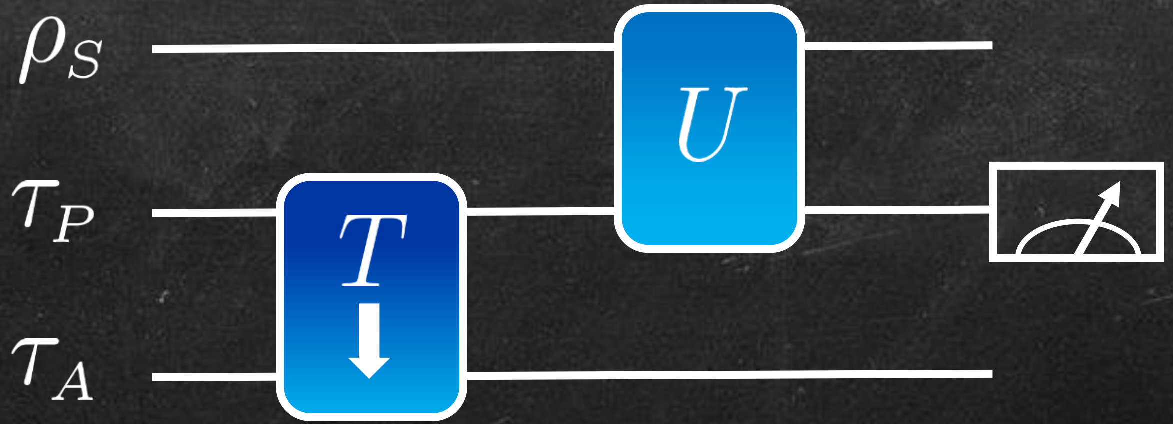
If we cannot have faithful measurements, what can we have?

➡ non-ideal but unbiased measurements

Given a measurement outcome: no deterministic statement about post-measurement system state possible

But: many outcomes give the correct statistics

General structure:



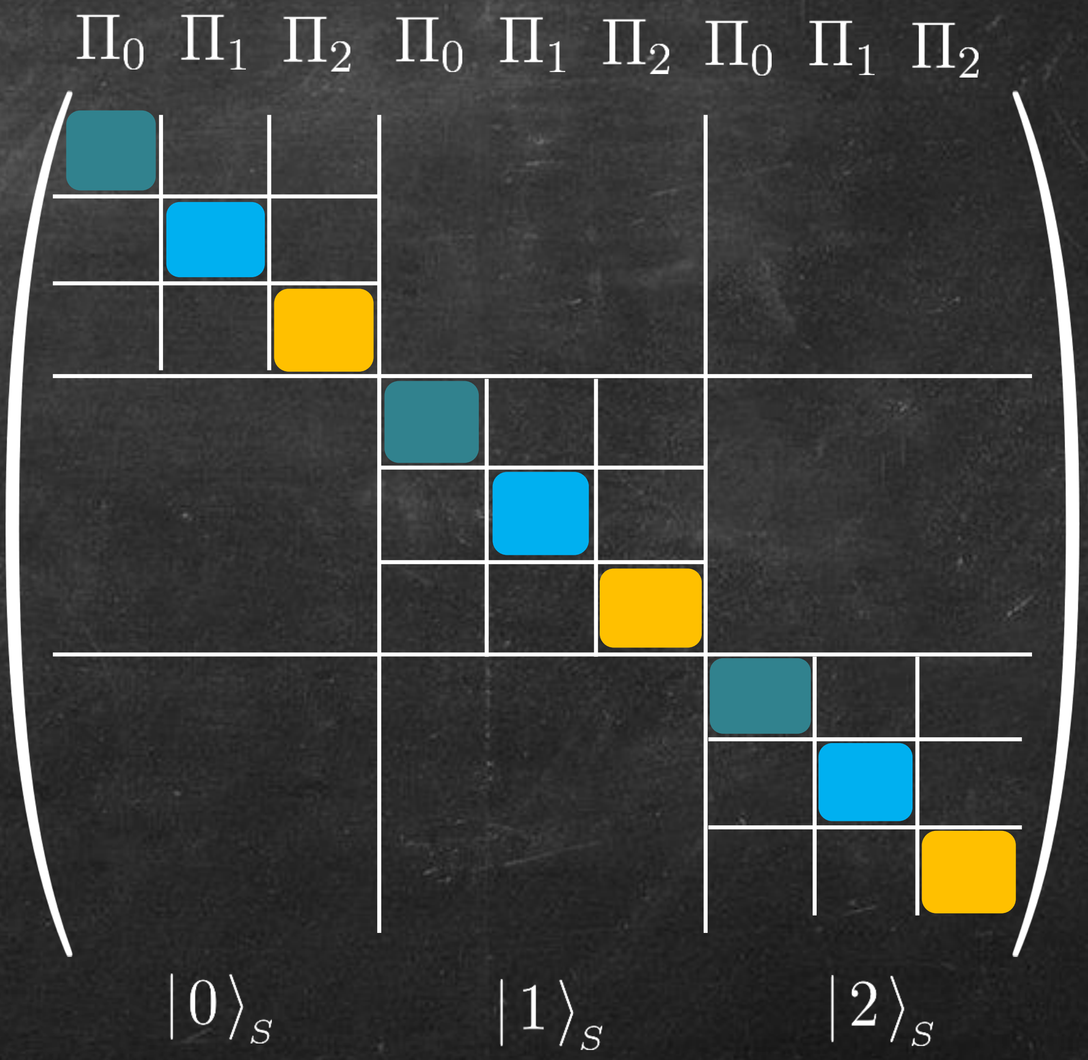
Unbiased measurements

Example: $d_S = 3$

$$\rho_S \otimes \tau_P$$

$$\begin{array}{ccc} \multicolumn{3}{c}{\begin{pmatrix} \rho_{00}\,\tau(\beta) & & \\ & \rho_{11}\,\tau(\beta) & \\ & & \rho_{22}\,\tau(\beta) \end{pmatrix}} \\ |0\rangle_S & |1\rangle_S & |2\rangle_S \end{array}$$

$\Rightarrow$



Notation $\quad \rho_{nn} = \langle n|\,\rho_S\,|n\rangle$

Maximally correlating unbiased measurements

Unbiasedness leaves some freedom

$$\tau(\beta) = \mathrm{diag}\{\tau_0, \tau_1, \tau_2, \ldots\}$$

Split weights in 3 parts: L, M, S

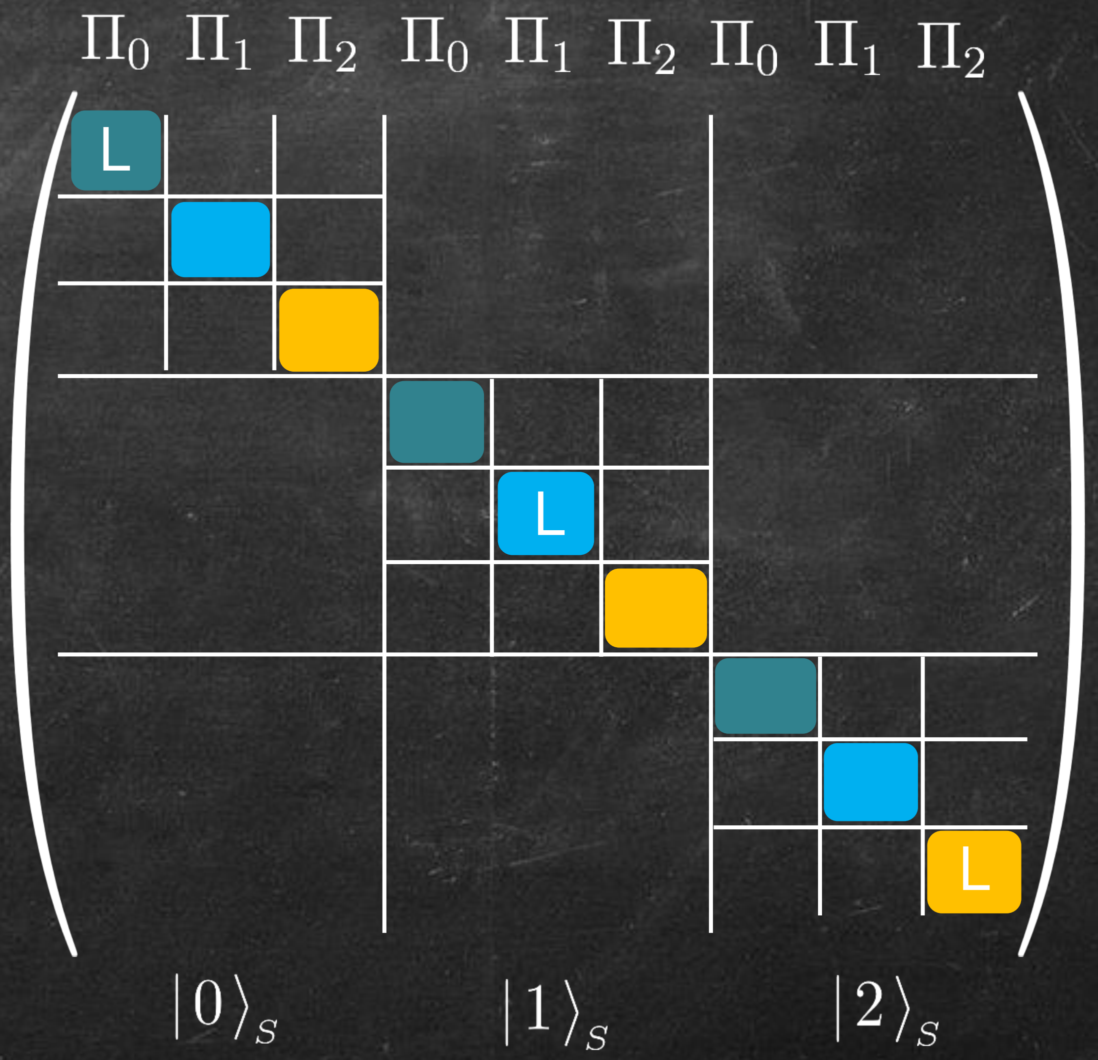
L: largest third of probabilities

M: middle third of probabilities

S: smallest third of probabilities

Maximally possible correlation:

Arrange L L L in $|i\rangle\langle i|_S \otimes \Pi_i$



Min. energy max. correlating unbiased measurements

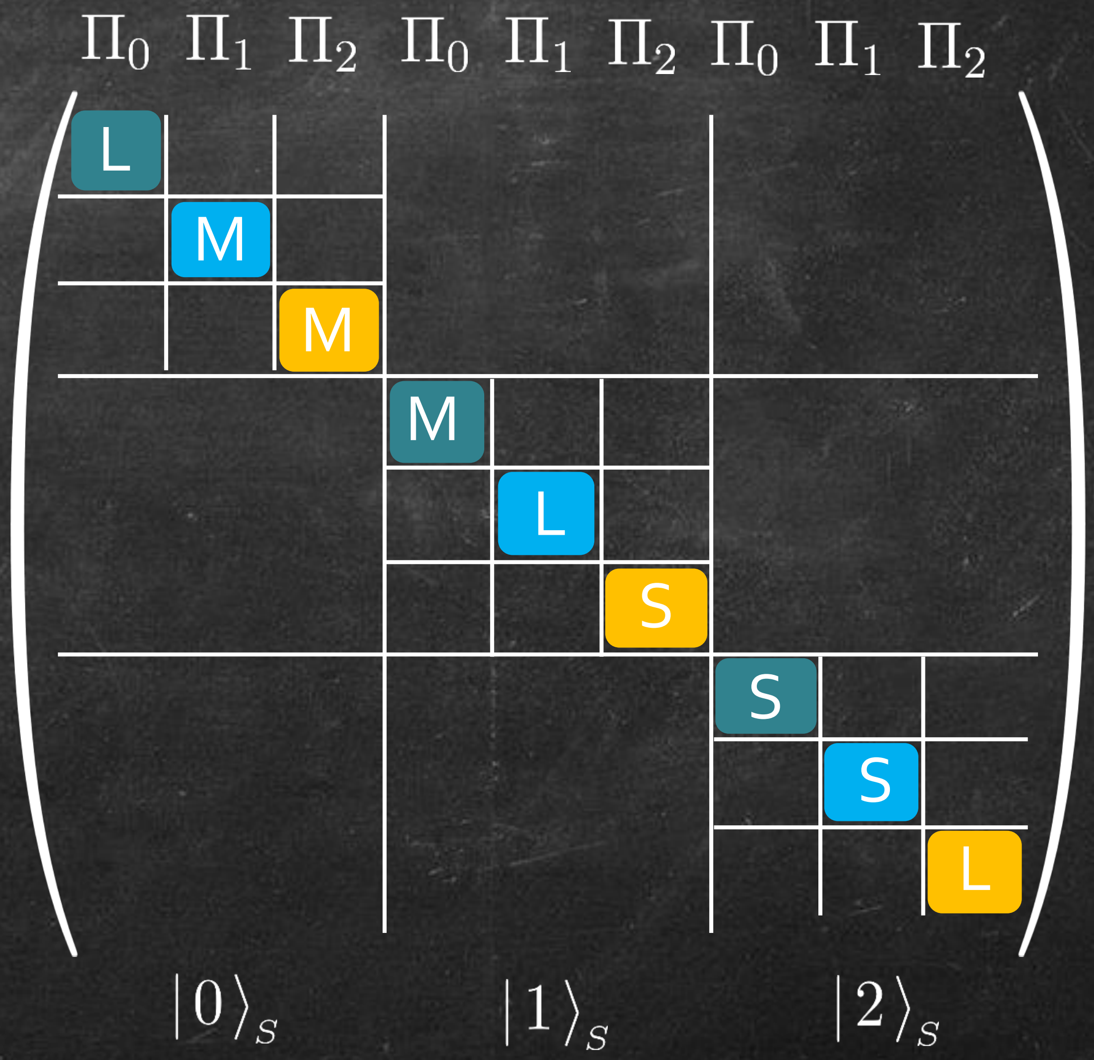
Further freedom to minimize energy

Pair largest remaining weights with smallest remaining energies
such that each color has each symbol (L, M, S) only once

Requires some information about system and pointer Hamiltonian

e.g., $H_S = H_P = \sum_n n\omega\,|n\rangle\langle n|$

(pointer also 3-dimensional)



Costs of cooling and correlating

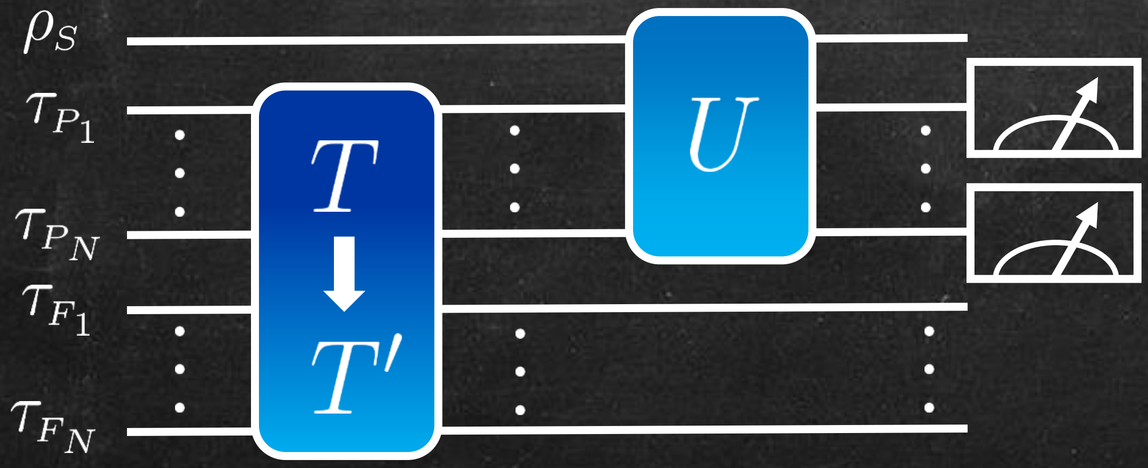
Given the Hamiltonian of system and pointer $H_S + H_P$

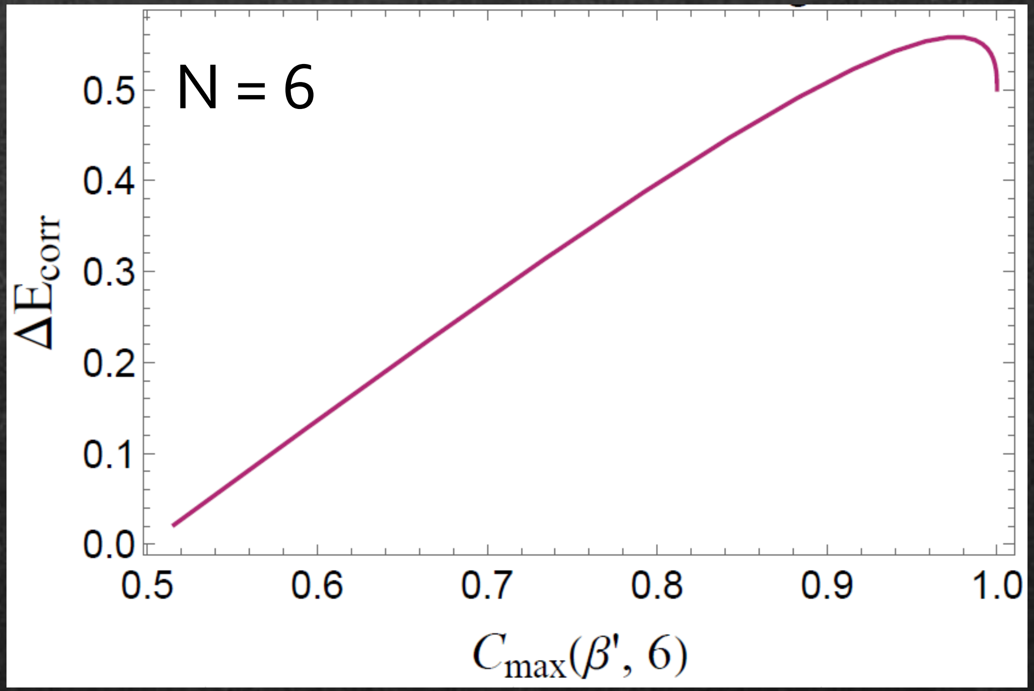
➡ Quantify the energy cost of maximally correlating ΔE_{corr}

Can also quantify cost of preparing the pointer, e.g., for cooling ΔE_{cool}

Example: 2-level system (qubit)

N-qubit pointer + N-qubit "fridge"





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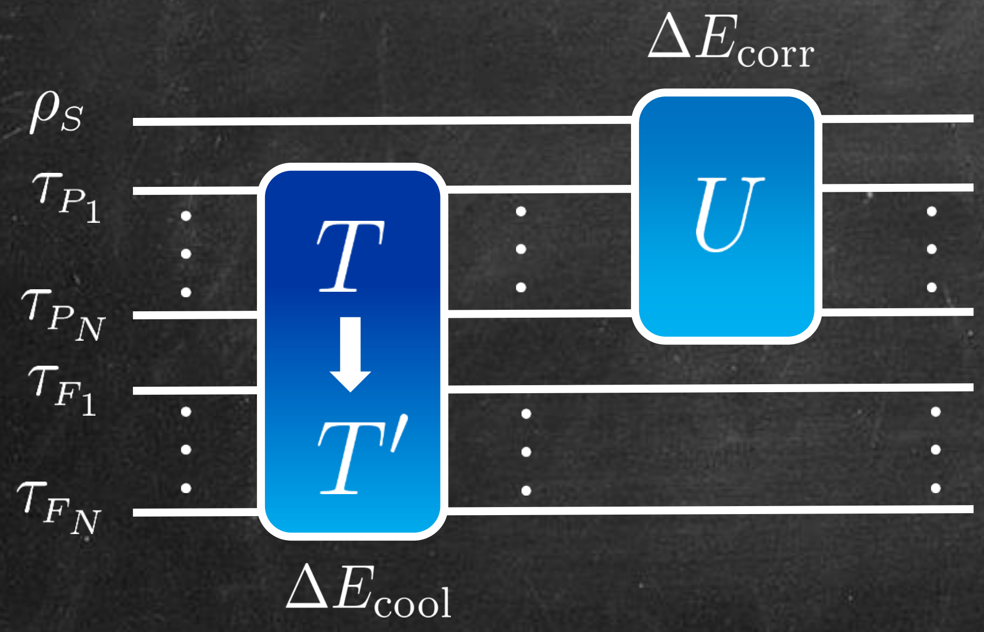
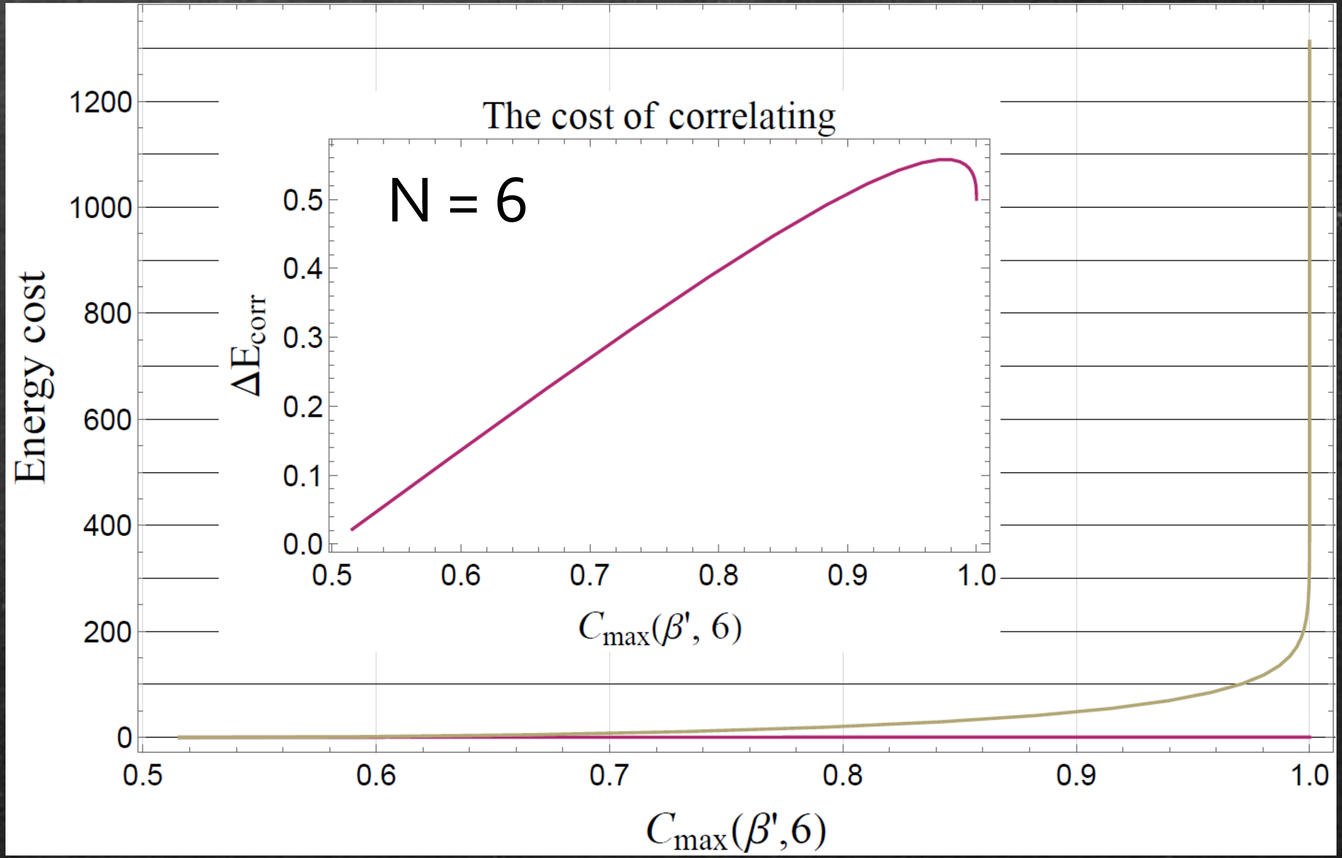




To achieve large correlation:

$$\boxed{\Delta E_{\mathrm{cool}} \gg \Delta E_{\mathrm{corr}}}$$

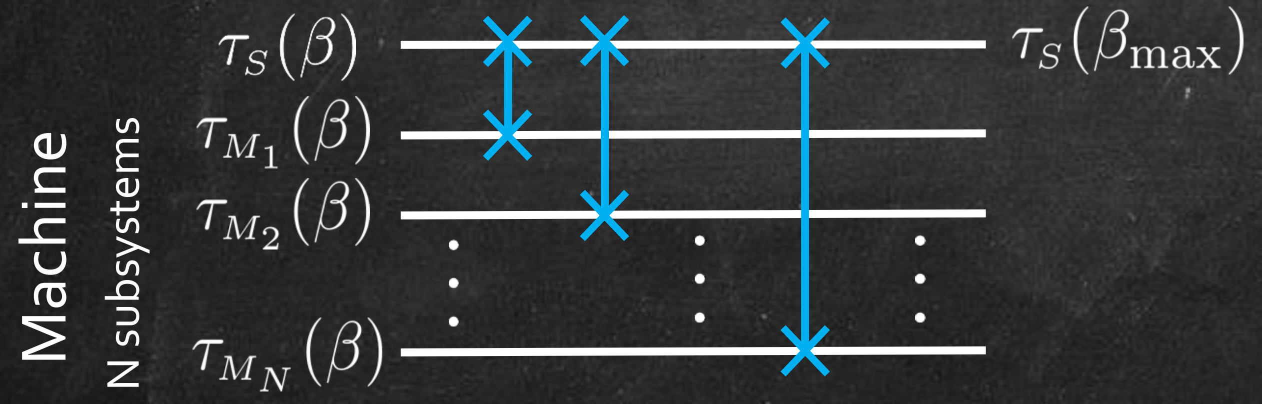
What are the resources for ground-state cooling?

Landauer: erasing a bit of information has energy cost $\Delta E \geq k_{\mathrm{B}} T \log(2)$

Nernst's unattainability principle (3rd law of thermodynamics):

Infinite resources are required to reach the ground state exactly

In the case of Landauer erasure: minimal energy but infinite time



Hamiltonian H_S

$$H_{M_n} = (1 + n\epsilon) H_S \qquad \epsilon = \frac{\beta_{\max} - \beta}{N\beta}$$

As $N \to \infty: \quad \Delta E \to k_{\mathrm{B}} T \, \Delta S_S$

Infinitely many "simple" steps OR one infinitely "complex" step

Rolf Landauer, *Irreversibility and Heat Generation in the Computing Process*, IBM J. Res. Dev. **5**, 183 (1961).

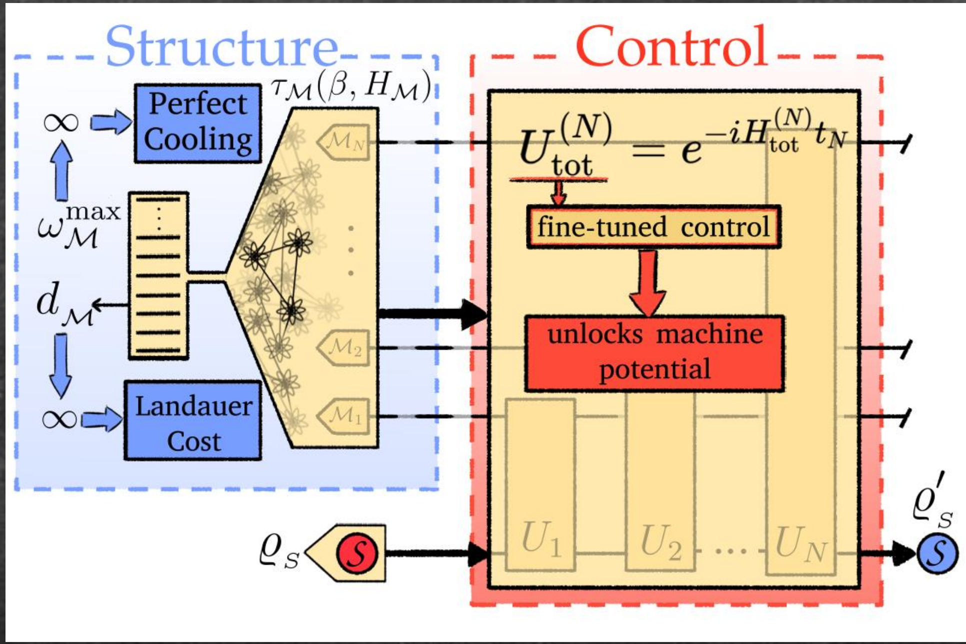
How can we quantify complexity?

Structural Complexity vs. Control Complexity

protocol-independent structural conditions fulfilled by the machine

For perfect cooling:

diverging energy gap in machine Hamiltonian



dynamic properties of the interaction that implements a given protocol

For perfect cooling:

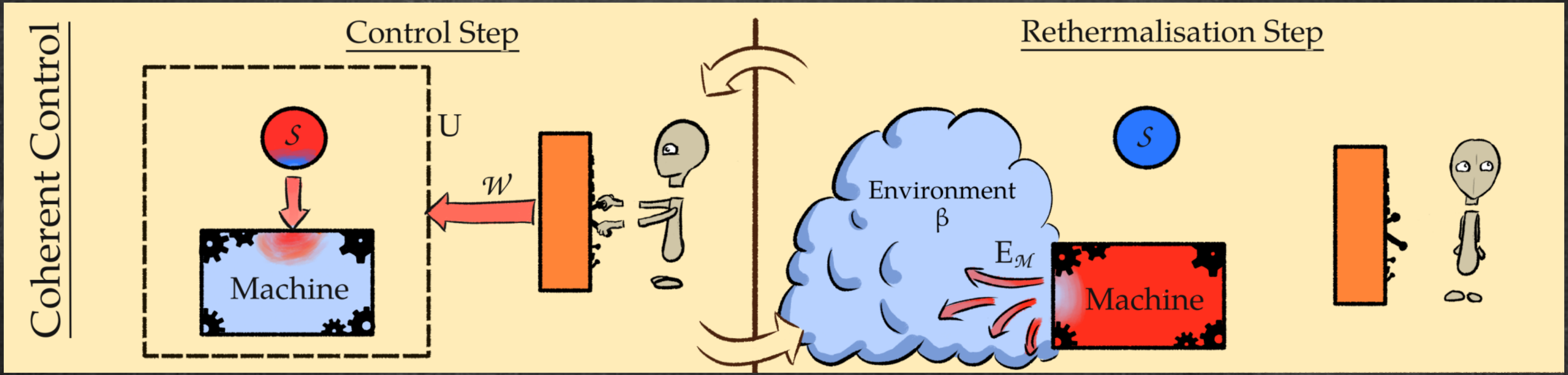
unitary representing cooling protocol must act nontrivially on ∞-dimensional subspace

effective dimension $d^{\text{eff}} := \min \dim(\mathcal{A}) : U_{SM} = U_{\mathcal{A}} \oplus \mathbb{1}_{\mathcal{A}^\perp}$

Philip Taranto, Faraj Bakhshinezhad, Andreas Bluhm, Ralph Silva, NF, Maximilian P. E. Lock, Giuseppe Vitagliano, Felix C. Binder, Tiago Debarba, Emanuel Schwarzhans, Fabien Clivaz, and Marcus Huber, Preprint arXiv:2106.05151 [quant-ph] (2021).

The Coherent-Control Paradigm

entails arbitrary unitaries on system and machine



Qudit

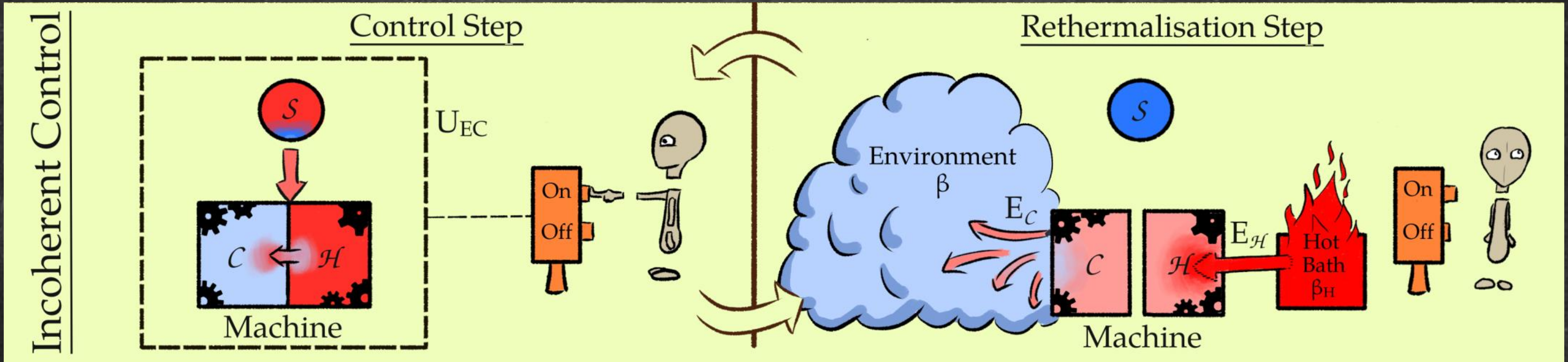
energy	time	control complexity
∞	1	$\frac{1}{2}d(d-1)$
Landauer	∞	$\frac{1}{2}d(d-1)$
Landauer	1	∞

Harm. Osci.

energy	time	control complexity
∞	1	∞ (Gaussian)
Landauer	∞	∞ (Gaussian)
>Landauer	∞	1 (Non-Gaussian)
Landauer	1	∞ (Gaussian)

The Incoherent-Control Paradigm

Hot bath and only energy-conserving unitaries



Perfect cooling is possible, provided either ∞ time or ∞ control complexity

But not at Landauer limit (for finite-temperature heat engine)

➡ Carnot-Landauer limit $\Delta F_{\mathcal{S}}(\beta) + (1 - \frac{\beta_{\mathcal{H}}}{\beta})\, \Delta E_{\mathcal{H}} \geq 0$

For finite time & control complexity, even ∞ energy not enough to reach ground state

The Thermodynamics Team

Yelena Guryanova, NF, and Marcus Huber,
Ideal projective measurements have infinite resource costs,
Quantum **4**, 222 (2020)

Philip Taranto, Faraj Bakhshinezhad, Andreas Bluhm, Ralph Silva, NF, Maximilian P. E. Lock, Giuseppe Vitagliano, Felix C. Binder, Tiago Debarba, Emanuel Schwarzhans, Fabien Clivaz, and Marcus Huber,
Landauer vs. Nernst: What is the True Cost of Cooling a Quantum System? Preprint arXiv:2106.05151 [quant-ph] (2021).

Thank you for your attention